EE 229

FALL SEMESTER

1974

I. VECTORS AND FIELDS

1. Vector Algebra and Cartesian Coordinate System
2. Cylindrical and spherical coordinate systems and Scalar and Vector Fields
3. Sinusoidally Time Varying Fields; Polarization
4. Electric Field
5. Magnetic Field

II. MAXWELL'S EQUATIONS IN INTEGRAL FORM

6. Line Integral
7. Surface Integral
8. Faraday's Law in Integral Form and Gauss' Law for the Magnetic Field
9. Ampere's Circuital Law in Integral Form
10. Gauss' Law for the Electric Field

III. MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

11. Faraday's Law in Differential Form
12. Ampere's Circuital Law in Differential Form
13. Curl and Stokes' Theorem
14. Gauss' Law in Differential Form; Divergence and Divergence Theorem

IV. WAVE PROPAGATION IN FREE SPACE

15. Fields due to Infinite Plane Current sheet
16. Uniform Plane Traveling Waves
17. Doppler Effect
18. Poynting Vector and Energy storage
19. Radiation Field of Antenna and Energy Stored in a Spherical charge Distribution

V. WAVE PROPAGATION IN MATERIAL MEDIA

20. Conductors and Dielectrics

21. Magnetic Materials

22. Uniform Plane Waves in Materials

23. Propagation Characteristics in Material Media

VI. TRANSMISSION LINES

24. Boundary Conditions on a Perfect Conductor Surface

25. Parallel Plate Transmission Line

26. Transmission Line with an Arbitrary Cross section

27. Short Circuited Transmission Line

28. Boundary Conditions at a Dielectric Discontinuity and Transmission Lines in Cascade

VII. WAVEGUIDES

29. Uniform Plane Wave Propagation in an Arbitrary Direction

30. TE Waves in a Parallel Plate Waveguide

31. Parallel Plate Waveguide Discontinuity

32. Rectangular waveguide and cavity resonator; optical waveguides

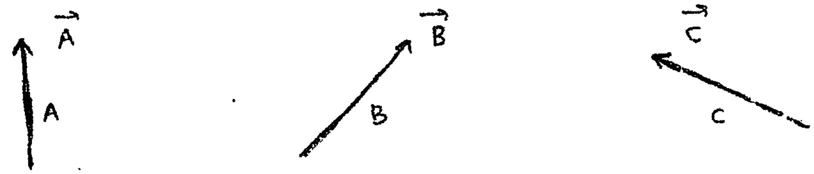
VIII. ANTENNAS

33. Hertzian dipole

34. Radiation resistance and directivity; arrays

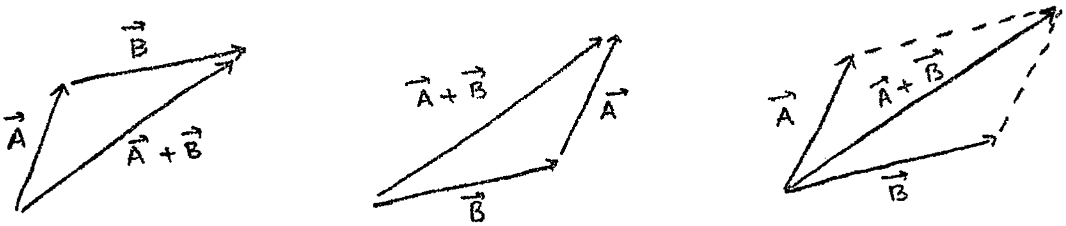
35. Half-wave dipole

Graphical representation of vectors:

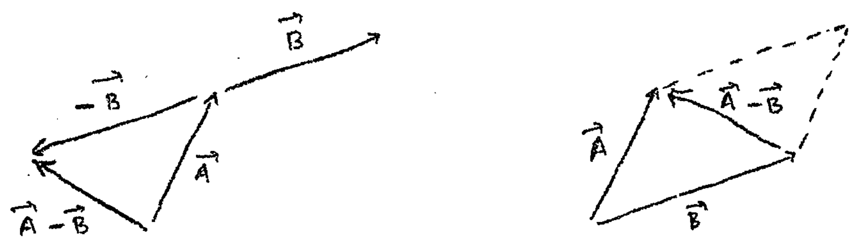


Simple Rules of Vector Algebra:

(a) Addition and subtraction of vectors.



$\vec{A}+\vec{B} = \vec{B}+\vec{A}$ commutative property is satisfied.



$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$

$\vec{A}+(\vec{B}+\vec{C}) = (\vec{A}+\vec{B})+\vec{C}$ Associative property

(b) Multiplication and Division by a Scalar

$$m\vec{A} = \begin{cases} \text{a vector whose magnitude is } |m|\,|\vec{A}| \\ \text{and whose direction is the direction of } \vec{A} \text{ if } m>0 \\ \text{and the direction of } (-\vec{A}) \text{ if } m<0 \end{cases}$$

$$(m+n)(\vec{A}+\vec{B}) = m(\vec{A}+\vec{B}) + n(\vec{A}+\vec{B})$$
$$= m\vec{A} + m\vec{B} + n\vec{A} + n\vec{B}$$

$$\frac{\vec{B}}{n} = \left(\frac{1}{n}\right)\vec{B}$$

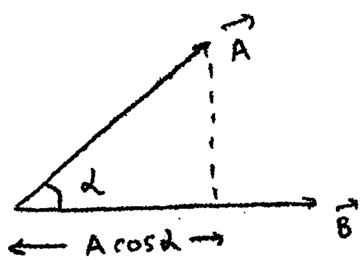
(c) Unit Vector.

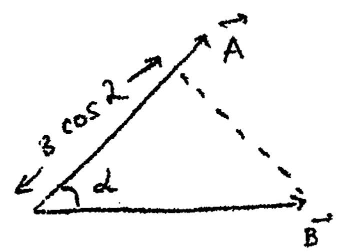
$$\vec{i}_A = \frac{\vec{A}}{|\vec{A}|} = \begin{cases} \text{a vector whose magnitude is } 1 \\ \text{and whose direction is that of } \vec{A} \end{cases}$$

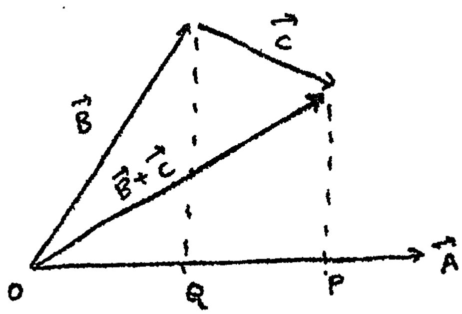
(d) Scalar or Dot product of two vectors:

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}| \cos \alpha\angle^{\vec{A}}_{\vec{B}} = AB\cos\alpha$$

$$\vec{A}\cdot\vec{B} = AB\cos\alpha = BA\cos\alpha = \vec{B}\cdot\vec{A}$$ commutative property holds.







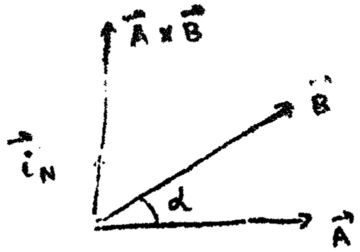
$$\vec{A}\cdot(\vec{B}+\vec{C}) = A\,(OP)$$
$$= A\,(OQ + QP)$$
$$= \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

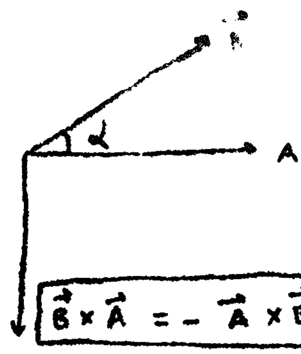
Distributive property holds.

(e) Vector or Cross Product of Two Vectors

$$\vec{A} \times \vec{B} = A B \sin \alpha_{\vec{A}}^{\vec{B}} \; \vec{i}_N$$

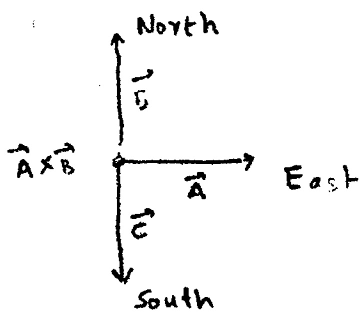
where $\vec{i}_N$ is the unit vector in the direction of advance of a right-hand screw as it is turned from $\vec{A}$ towards $\vec{B}$ through α.





$$\boxed{\vec{B} \times \vec{A} = - \vec{A} \times \vec{B}}$$

Commutative law does not hold.



$$(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$$

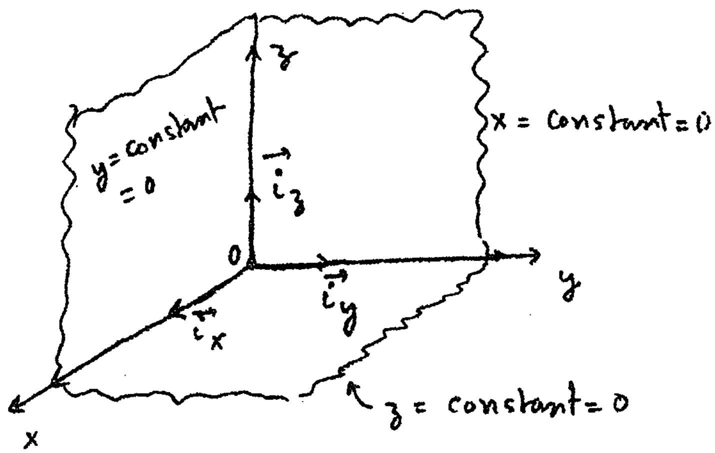
Associative law does not hold.

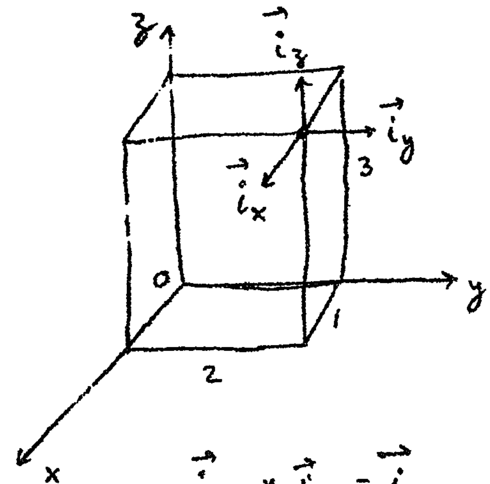
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad \text{Distributive law}$$

(f) Defining Unit Vector Using Cross Product operation

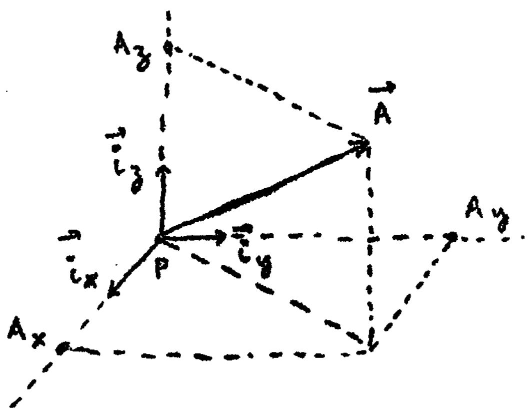
$$\vec{i}_N = \frac{\vec{A} \times \vec{B}}{A B \sin \alpha}$$

Cartesian Coordinate System:





$$\vec{i}_x \times \vec{i}_y = \vec{i}_z$$
$$\vec{i}_y \times \vec{i}_z = \vec{i}_x$$
$$\vec{i}_z \times \vec{i}_x = \vec{i}_y$$



$$\vec{A} = A_x \vec{i}_x + A_y \vec{i}_y + A_z \vec{i}_z$$
$$= (\vec{A}\cdot\vec{i}_x)\vec{i}_x + (\vec{A}\cdot\vec{i}_y)\vec{i}_y + (\vec{A}\cdot\vec{i}_z)\vec{i}_z$$

Consider three vectors at a point:

$$\vec{A} = A_x \vec{i}_x + A_y \vec{i}_y + A_z \vec{i}_z$$
$$\vec{B} = B_x \vec{i}_x + B_y \vec{i}_y + B_z \vec{i}_z$$
$$\vec{C} = C_x \vec{i}_x + C_y \vec{i}_y + C_z \vec{i}_z$$

Then,

(a) Equality of vectors:

Two vectors are equal if and only if their respective components are equal, i.e. $B_x = A_x$, $B_y = A_y$, and $B_z = A_z$

(b) Magnitude of a vector:

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

(c) Addition and subtraction of vectors

$$\vec{A} + \vec{B} = (A_x + B_x)\,\vec{i}_x + (A_y + B_y)\,\vec{i}_y + (A_z + B_z)\,\vec{i}_z$$

$$\vec{B} - \vec{C} = (B_x - C_x)\,\vec{i}_x + (B_y - C_y)\,\vec{i}_y + (B_z - C_z)\,\vec{i}_z$$

(d) Multiplication and Division by Scalar

$$m\vec{A} = mA_x\,\vec{i}_x + mA_y\,\vec{i}_y + mA_z\,\vec{i}_z$$

$$\frac{\vec{B}}{n} = \frac{B_x}{n}\,\vec{i}_x + \frac{B_y}{n}\,\vec{i}_y + \frac{B_z}{n}\,\vec{i}_z$$

(e) Unit vector along $\vec{A}$:

$$\vec{i}_A = \frac{A_x\vec{i}_x + A_y\vec{i}_y + A_z\vec{i}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

(f) Dot product of two vectors:

$$\vec{A}\cdot\vec{B} = (A_x\vec{i}_x + A_y\vec{i}_y + A_z\vec{i}_z)\cdot(B_x\vec{i}_x + B_y\vec{i}_y + B_z\vec{i}_z)$$

$$= A_xB_x + A_yB_y + A_zB_z$$

(g) Cross product of two vectors:

$$\vec{A}\times\vec{B} = (A_x\vec{i}_x + A_y\vec{i}_y + A_z\vec{i}_z)\times(B_x\vec{i}_x + B_y\vec{i}_y + B_z\vec{i}_z)$$

$$= A_xB_y\vec{i}_z - A_xB_z\vec{i}_y - A_yB_x\vec{i}_z + A_yB_z\vec{i}_x + A_zB_x\vec{i}_y - A_zB_y\vec{i}_x$$

$$= (A_yB_z - A_zB_y)\,\vec{i}_x + (A_zB_x - A_xB_z)\,\vec{i}_y + (A_xB_y - A_yB_x)\,\vec{i}_z$$

$$= \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example: let $\vec{A} = 2\vec{i}_x - \vec{i}_z$

$\vec{B} = 2\vec{i}_x - \vec{i}_y + 2\vec{i}_z$

$\vec{C} = 2\vec{i}_x - 3\vec{i}_y + \vec{i}_z$

Then

(a) $\vec{A} + \vec{B} = 4\vec{i}_x - \vec{i}_y + \vec{i}_z$

(b) $\vec{B} - \vec{C} = 2\vec{i}_y + \vec{i}_z$

(c) $\vec{A} + \vec{B} - \vec{C} = \vec{A} + (\vec{B} - \vec{C}) = 2\vec{i}_x + 2\vec{i}_y$

(d) $|\vec{B}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$

(e) $\vec{i}_B = \dfrac{2\vec{i}_x - \vec{i}_y + 2\vec{i}_z}{3} = \frac{2}{3}\vec{i}_x - \frac{1}{3}\vec{i}_y + \frac{2}{3}\vec{i}_z$

(f) $\vec{A} \cdot \vec{B} = (2)(2) + (0)(-1) + (-1)(2) = 4 + 0 - 2 = 2$

(g) cosine of the angle between $\vec{A}$ and $\vec{B}$ = $\dfrac{\vec{A} \cdot \vec{B}}{AB} = \dfrac{2}{3\sqrt{5}}$

(h) $\vec{B} \times \vec{C} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 5\vec{i}_x + 2\vec{i}_y - 4\vec{i}_z$

(i) sine of the angle between $\vec{B}$ and $\vec{C}$ = $\dfrac{|\vec{B} \times \vec{C}|}{BC} = \dfrac{\sqrt{45}}{3\sqrt{14}} = \sqrt{\dfrac{5}{14}}$.

(j) $\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ 2 & 0 & -1 \\ 5 & 2 & -4 \end{vmatrix} = -2\vec{i}_x + 3\vec{i}_y + 4\vec{i}_z$

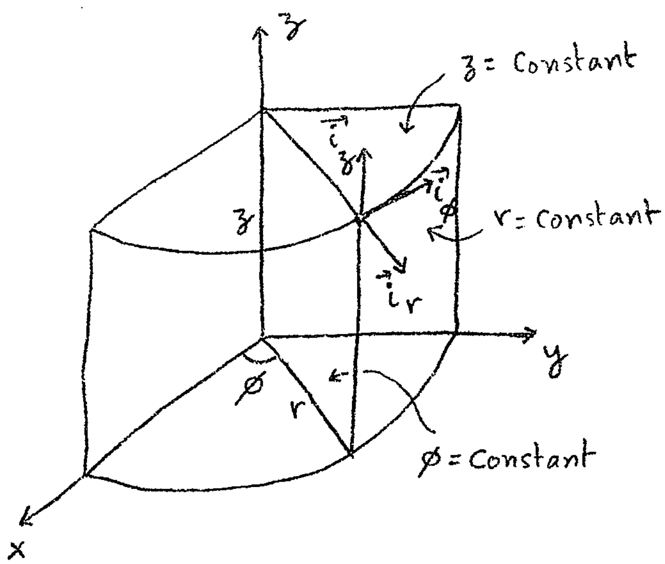
(k) A vector perpendicular to $\vec{B}$ is given by

$$\vec{D} = \vec{B} \times \vec{i}_x = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ 2 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\vec{i}_y + \vec{i}_z$$

Verification: $\vec{B} \cdot \vec{D} = 2(0) + (-1)(2) + (2)(1) = 0 - 2 + 2 = 0$.

Cylindrical and Spherical Coordinate Systems:

Cylindrical



r, ϕ, z

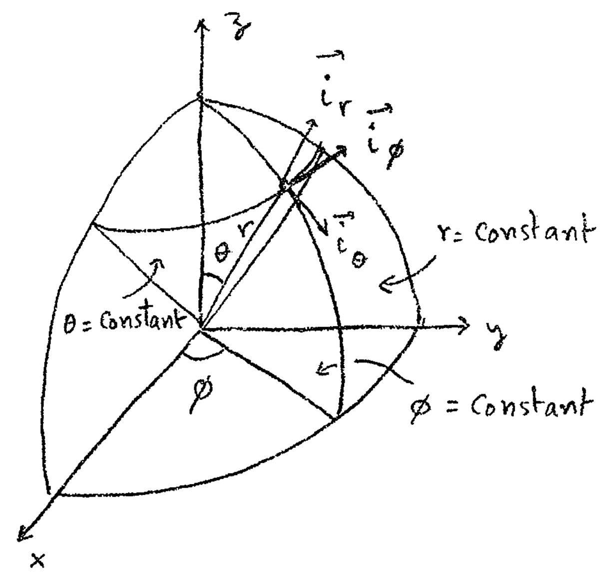
cylinder, plane, plane

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ z = z \end{cases}$$

$$\begin{cases} \vec{i}_r = \cos\phi\,\vec{i}_x + \sin\phi\,\vec{i}_y \\ \vec{i}_\phi = -\sin\phi\,\vec{i}_x + \cos\phi\,\vec{i}_y \\ \vec{i}_z = \vec{i}_z \end{cases}$$

Spherical



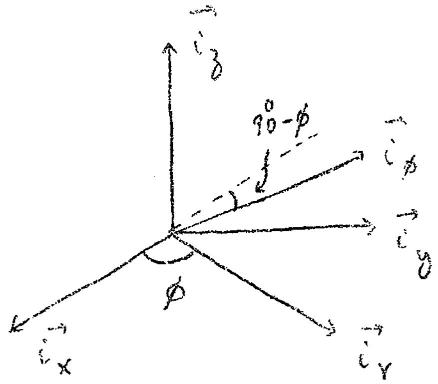
r, θ, ϕ

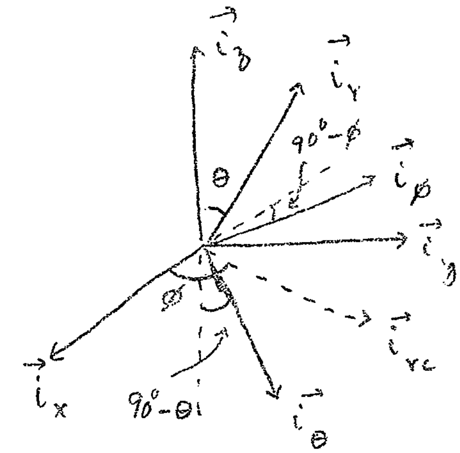
sphere, cone, plane

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} \\ \phi = \tan^{-1}\dfrac{y}{x} \end{cases}$$

$$\begin{cases} \vec{i}_r = \sin\theta(\cos\phi\,\vec{i}_x + \sin\phi\,\vec{i}_y) + \cos\theta\,\vec{i}_z \\ \vec{i}_\theta = \cos\theta\,(\cos\phi\,\vec{i}_x + \sin\phi\,\vec{i}_y) - \sin\theta\,\vec{i}_z \\ \vec{i}_\phi = -\sin\phi\,\vec{i}_x + \cos\phi\,\vec{i}_y \end{cases}$$





$$\vec{i}_r \cdot \vec{i}_x = \cos\phi$$

$$\vec{i}_r \cdot \vec{i}_y = \sin\phi$$

$$\vec{i}_r \cdot \vec{i}_z = 0$$

$$\vec{i}_\phi \cdot \vec{i}_x = -\cos(90^\circ-\phi)$$
$$= -\sin\phi$$

$$\vec{i}_\phi \cdot \vec{i}_y = \sin(90^\circ-\phi)$$
$$= \cos\phi$$

$$\vec{i}_\phi \cdot \vec{i}_z = 0$$

$$\vec{i}_r \cdot \vec{i}_x = \sin\theta\cos\phi$$

$$\vec{i}_r \cdot \vec{i}_y = \sin\theta\sin\phi$$

$$\vec{i}_r \cdot \vec{i}_z = \cos\theta$$

$$\vec{i}_\theta \cdot \vec{i}_x = \sin(90^\circ-\theta)\cos\phi$$
$$= \cos\theta\cos\phi$$

$$\vec{i}_\theta \cdot \vec{i}_y = \sin(90^\circ-\theta)\sin\phi$$
$$= \cos\theta\sin\phi$$

$$\vec{i}_\theta \cdot \vec{i}_z = -\cos(90^\circ-\theta)$$
$$= -\sin\theta$$

$$\vec{i}_\phi \cdot \vec{i}_x = -\sin\phi$$

$$\vec{i}_\phi \cdot \vec{i}_y = \cos\phi$$

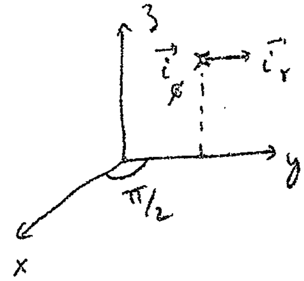
$$\vec{i}_\phi \cdot \vec{i}_z = 0$$

Example:

(a) $\vec{i}_r + \vec{i}_\phi + 3\vec{i}_z$ at $(2, \frac{\pi}{2}, 3)$

$$= \left(\cos\frac{\pi}{2}\,\vec{i}_x + \sin\frac{\pi}{2}\,\vec{i}_y\right) + \left(-\sin\frac{\pi}{2}\,\vec{i}_x + \cos\frac{\pi}{2}\,\vec{i}_y\right) + 3\vec{i}_z$$

$$= \vec{i}_y - \vec{i}_x + 3\vec{i}_z$$

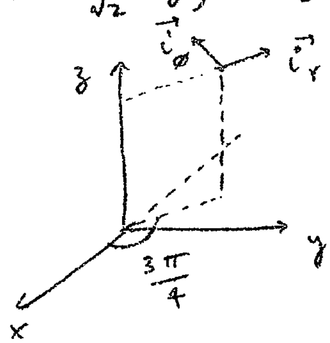


(b) $\vec{i}_r + \vec{i}_\phi + 3\vec{i}_z$ at $(3.6, \frac{3\pi}{4}, 9.4)$

$$= \left(\cos\frac{3\pi}{4}\,\vec{i}_x + \sin\frac{3\pi}{4}\,\vec{i}_y\right) + \left(-\sin\frac{3\pi}{4}\,\vec{i}_x + \cos\frac{3\pi}{4}\,\vec{i}_y\right) + 3\vec{i}_z$$

$$= \left(-\frac{1}{\sqrt{2}}\,\vec{i}_x + \frac{1}{\sqrt{2}}\,\vec{i}_y\right) + \left(-\frac{1}{\sqrt{2}}\,\vec{i}_x - \frac{1}{\sqrt{2}}\,\vec{i}_y\right) + 3\vec{i}_z$$

$$= -\sqrt{2}\,\vec{i}_x + 3\vec{i}_z$$



(c) $\sqrt{2}\,\vec{i}_r + 3\vec{i}_z$ at $(3.6, \frac{3\pi}{4}, 9.4)$

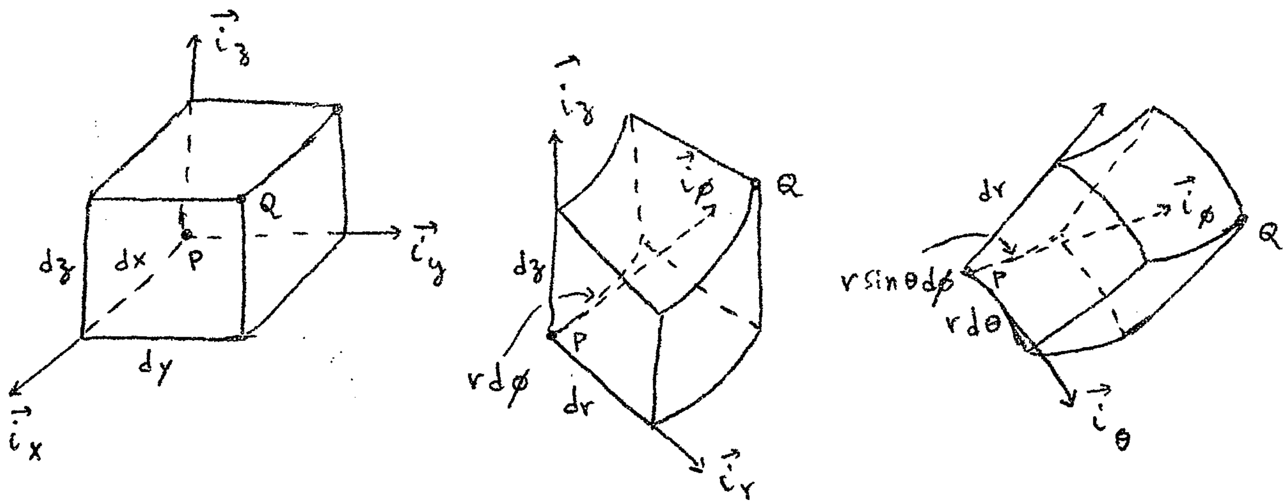
$$= \sqrt{2}\left(\cos\frac{3\pi}{4}\,\vec{i}_x + \sin\frac{3\pi}{4}\,\vec{i}_y\right) + 3\vec{i}_z$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}}\,\vec{i}_x + \frac{1}{\sqrt{2}}\,\vec{i}_y\right) + 3\vec{i}_z$$

$$= -\vec{i}_x + \vec{i}_y + 3\vec{i}_z$$

Conclusion: (b) and (a) are not equal but (c) and (a) are equal.

Differential lengths, surfaces and volumes:



Differential lengths:

$dx\,\vec{i}_x$, $dy\,\vec{i}_y$, $dz\,\vec{i}_z$ | $dr\,\vec{i}_r$, $r\,d\phi\,\vec{i}_\phi$, $dz\,\vec{i}_z$ | $dr\,\vec{i}_r$, $r\,d\theta\,\vec{i}_\theta$, $r\sin\theta\,d\phi\,\vec{i}_\phi$

$$\vec{PQ} = d\vec{l} = dx\,\vec{i}_x + dy\,\vec{i}_y + dz\,\vec{i}_z$$

$$d\vec{l} = dr\,\vec{i}_r + r\,d\phi\,\vec{i}_\phi + dz\,\vec{i}_z$$

$$d\vec{l} = dr\,\vec{i}_r + r\,d\theta\,\vec{i}_\theta + r\sin\theta\,d\phi\,\vec{i}_\phi$$

Differential surfaces:

$\pm\,dx\,dy\,\vec{i}_z$	$\pm\,r\,d\phi\,dr\,\vec{i}_z$	$\pm\,r\,dr\,d\theta\,\vec{i}_\phi$
$\pm\,dy\,dz\,\vec{i}_x$	$\pm\,r\,d\phi\,dz\,\vec{i}_r$	$\pm\,r\sin\theta\,d\theta\,d\phi\,\vec{i}_r$
$\pm\,dz\,dx\,\vec{i}_y$	$\pm\,dr\,dz\,\vec{i}_\phi$	$\pm\,r\,dr\,d\theta\,\vec{i}_\phi$

Differential volumes:

$dx\,dy\,dz$ | $r\,dr\,d\phi\,dz$ | $r^2\sin\theta\,dr\,d\theta\,d\phi$

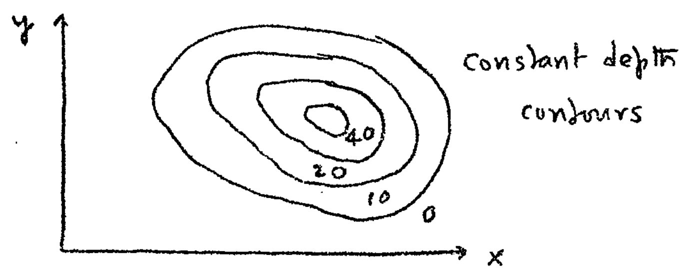
Scalar and Vector Fields

A mathematical function or a graphical sketch constructed so as to describe the variation of a quantity in a given region is said to represent the "field" of that quantity associated with that region.

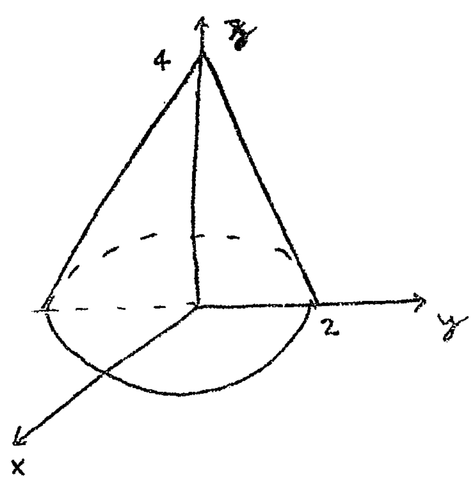
Examples of scalar fields:

Temparature in a room as a function of position and time ; $T(x, y, z; t)$

Depth of water in a lake as a function position on the surface; $d(x, y)$



Height of points on the surface of a cone above its base



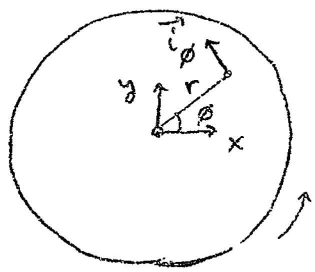
$$h = 4 - 2\sqrt{x^2 + y^2}$$

Vector fields

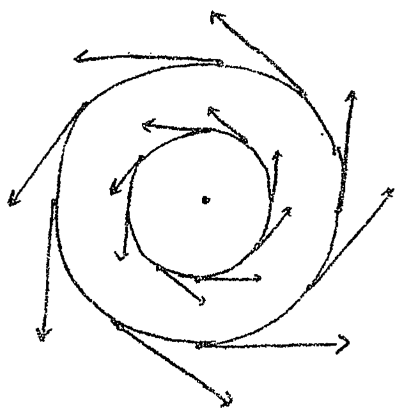
Need to describe not only how the magnitude varies but also how the direction of the vector varies as a function of the independent variables.

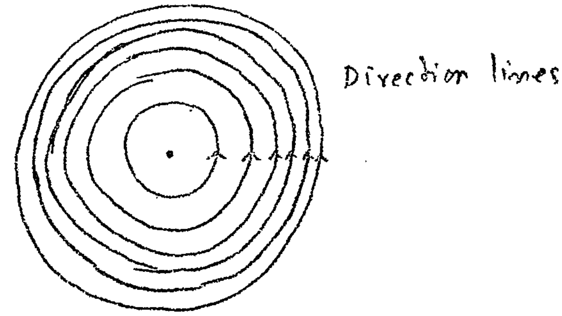
Example of Vector Field:

Velocity vector field associated with points on a circular rotating disk.



$$\vec{v} = r\omega\,\vec{i}_\phi$$
$$= r\omega\,(-\sin\phi\,\vec{i}_x + \cos\phi\,\vec{i}_y)$$
$$= \omega\,(-y\,\vec{i}_x + x\,\vec{i}_y)$$





Lines indicate direction of vector.
Density of lines indicate variation of the magnitude of the vector.

Sinusoidally Time Varying Fields

Scalar fields:

Example: $V(t) = V_0 \cos(\omega t + \phi)$

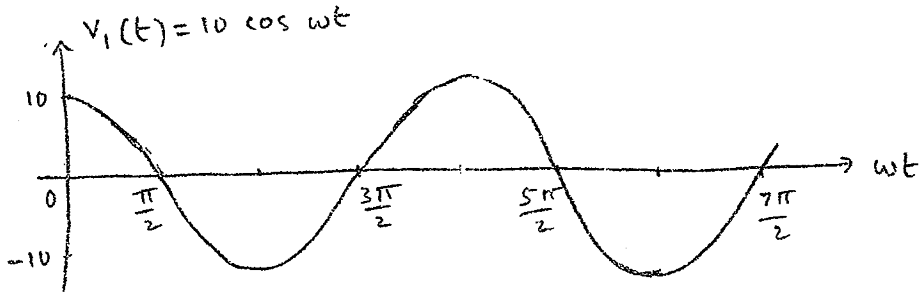
$\omega = 2\pi f$ = radian frequency

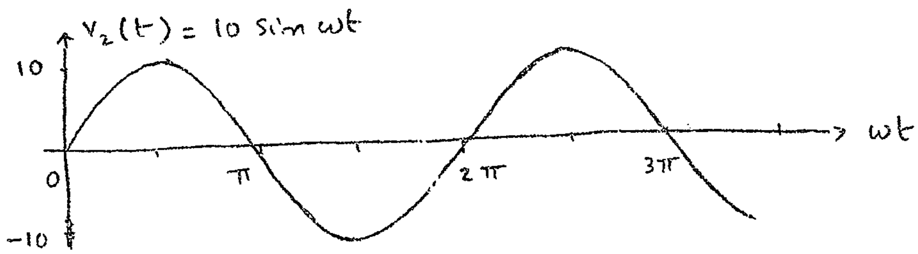
f = frequency

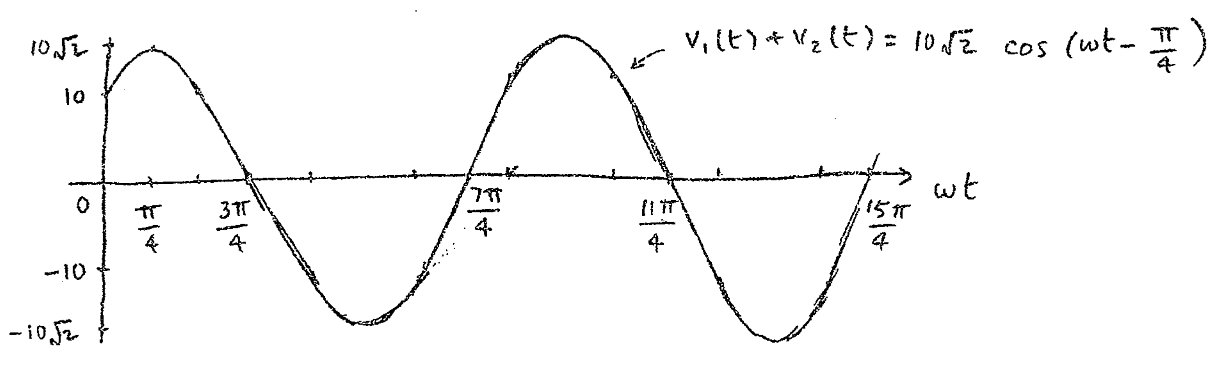
$\omega t + \phi$ = phase

ϕ = phase at $t = 0$.

Addition of sinusoidally time varying scalar fields:







Let $10 \cos \omega t + 10 \sin \omega t = A \cos(\omega t + \phi)$

$$= A \cos \omega t \cos \phi - A \sin \omega t \sin \phi$$

Then $A \cos \phi = 10$ and $A \sin \phi = -10$

$$A^2 = 10^2 + (-10)^2 = 200 \quad \text{and} \quad A = 10\sqrt{2}$$

$$\cos \phi = \frac{10}{A} = \frac{1}{\sqrt{2}}, \quad \sin \phi = \frac{-10}{A} = -\frac{1}{\sqrt{2}}, \quad \phi = -\frac{\pi}{4}$$

$$\therefore \; 10 \cos \omega t + 10 \sin \omega t = 10\sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right)$$

Alternatively, one can use the <u>phasor approach</u>:

$$10 \cos \omega t + 10 \sin \omega t$$

$$= 10 \cos \omega t + 10 \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$= \mathrm{Re}\left[10 e^{j\omega t}\right] + \mathrm{Re}\left[10 e^{j(\omega t - \frac{\pi}{2})}\right]$$

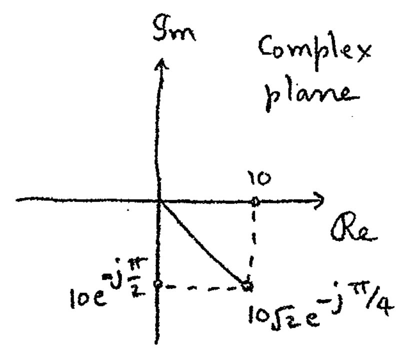
$$= \mathrm{Re}\left[10 e^{j\omega t}\right] + \mathrm{Re}\left[10 e^{-j\frac{\pi}{2}} e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[(10 + 10 e^{-j\pi/2}) e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[(10\sqrt{2}\, e^{-j\pi/4}) e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[10\sqrt{2}\, e^{j(\omega t - \pi/4)}\right]$$

$$= 10\sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right)$$

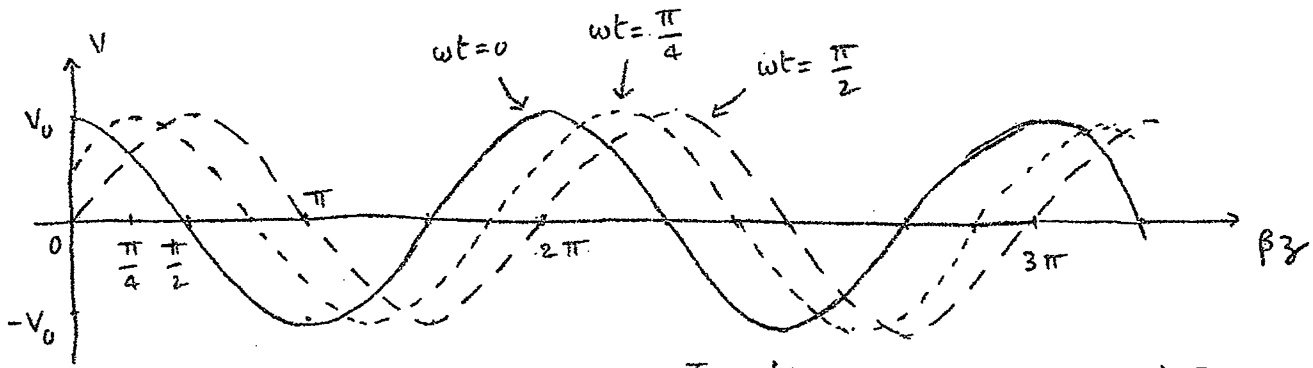


$$10 \cos \omega t \quad + \quad 10 \sin \omega t$$
$$\downarrow \qquad\qquad \downarrow$$
$$10 \cos \omega t \quad + \quad 10 \cos\left(\omega t - \frac{\pi}{2}\right)$$
$$\downarrow \qquad\qquad \downarrow$$
$$10 e^{j0} \quad + \quad 10 e^{-j\frac{\pi}{2}}$$

Phasors

$$= 10\sqrt{2}\, e^{-j\frac{\pi}{4}}$$
$$\downarrow$$
$$10\sqrt{2} \cos\left(\omega t - \frac{\pi}{4}\right)$$

$V(z,t) = V_0 \cos(\omega t - \beta z)$



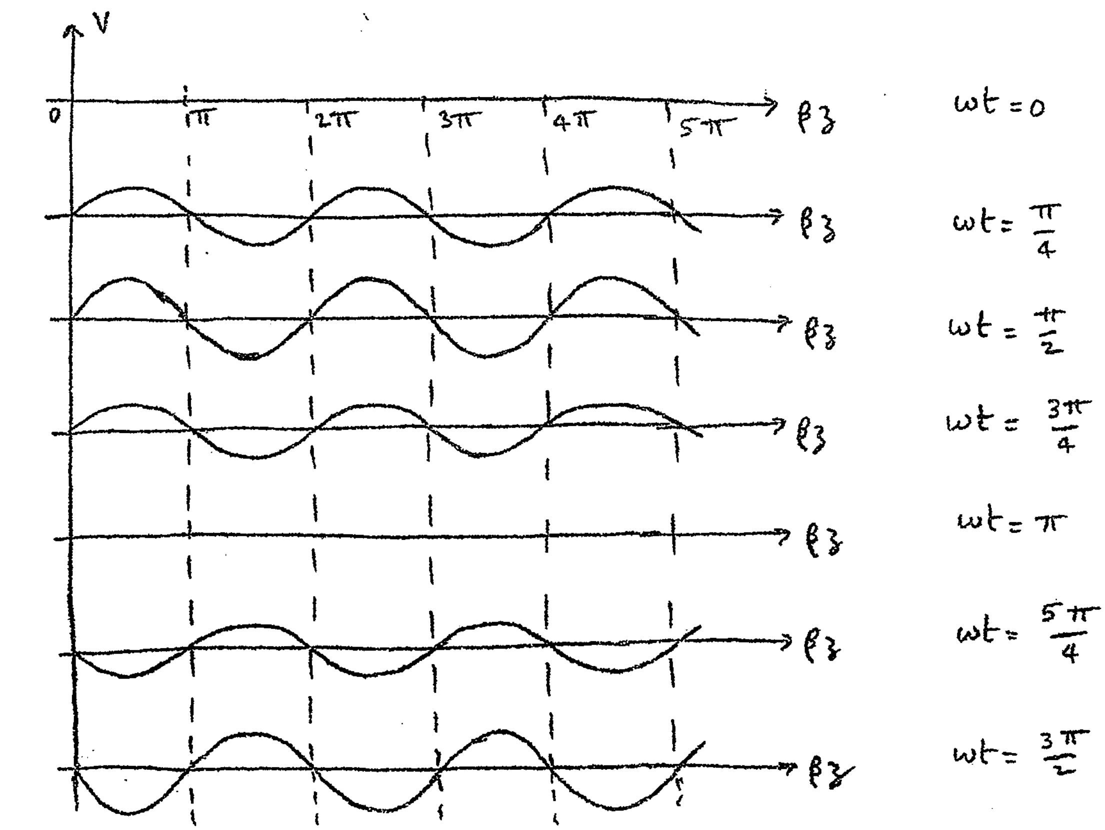
Traveling wave progressing in the z direction

In a time of $\frac{\pi}{4\omega}$, distance traveled $= \frac{\pi}{4\beta}$

$\therefore$ Velocity of propagation $= \dfrac{\pi/4\beta}{\pi/4\omega} = \dfrac{\omega}{\beta}$

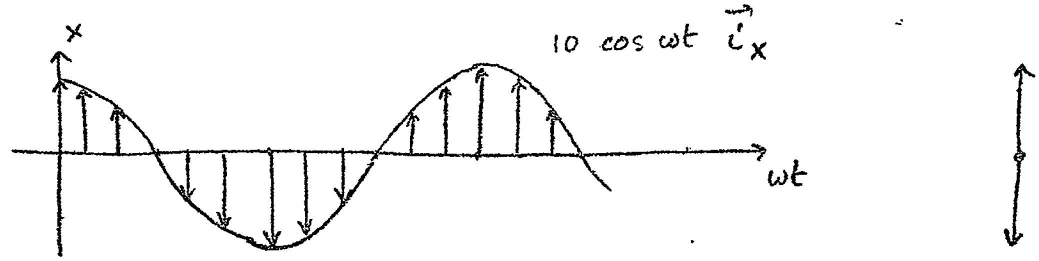
Example : $10 \cos(2\pi \times 10^8 t - 0.1\pi z) \rightarrow v = \dfrac{2\pi \times 10^8}{0.1\pi} = 2 \times 10^9$ m/sec.

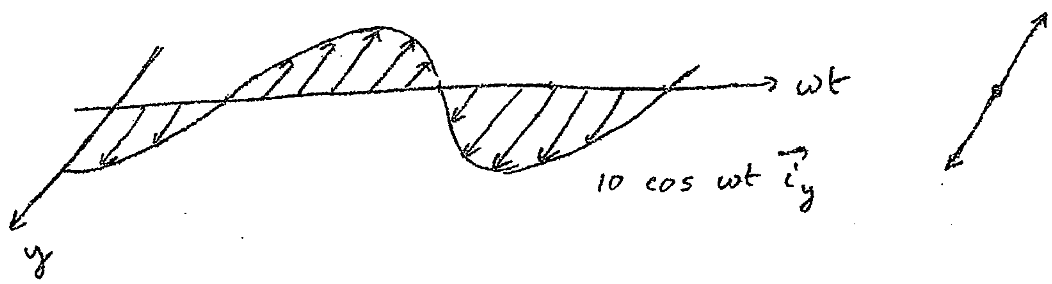
$V(z,t) = V_0 \sin\beta z \sin\omega t$



Standing waves

Vector fields:





Addition of sinusoidally time varying vector fields:

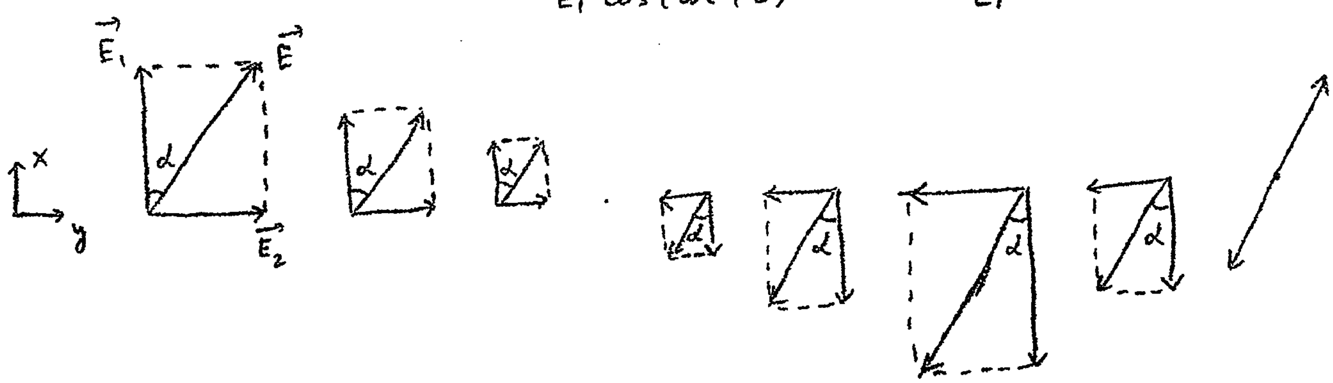
Linear Polarization:

$$\vec{E} = E_1 \cos(\omega t + \theta)\, \vec{i}_x + E_2 \cos(\omega t + \theta)\, \vec{i}_y$$

The two fields have different directions but are in phase

$$|\vec{E}| = \left[E_1^2 \cos^2(\omega t+\theta) + E_2^2 \cos^2(\omega t+\theta)\right]^{1/2} = \sqrt{E_1^2+E_2^2}\, \cos(\omega t+\theta)$$

Direction of $\vec{E}$: $\alpha = \tan^{-1} \dfrac{E_2 \cos(\omega t+\theta)}{E_1 \cos(\omega t+\theta)} = \tan^{-1} \dfrac{E_2}{E_1}$.



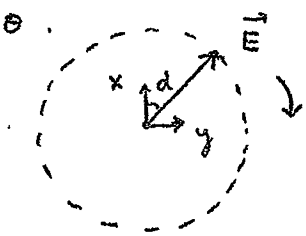
Circular Polarization:

$$\vec{E} = E_0 \cos(\omega t + \theta)\, \vec{i}_x + E_0 \sin(\omega t + \theta)\, \vec{i}_y$$
$$= E_0 \cos(\omega t + \theta)\, \vec{i}_x + E_0 \cos\left(\omega t + \theta - \frac{\pi}{2}\right) \vec{i}_y$$

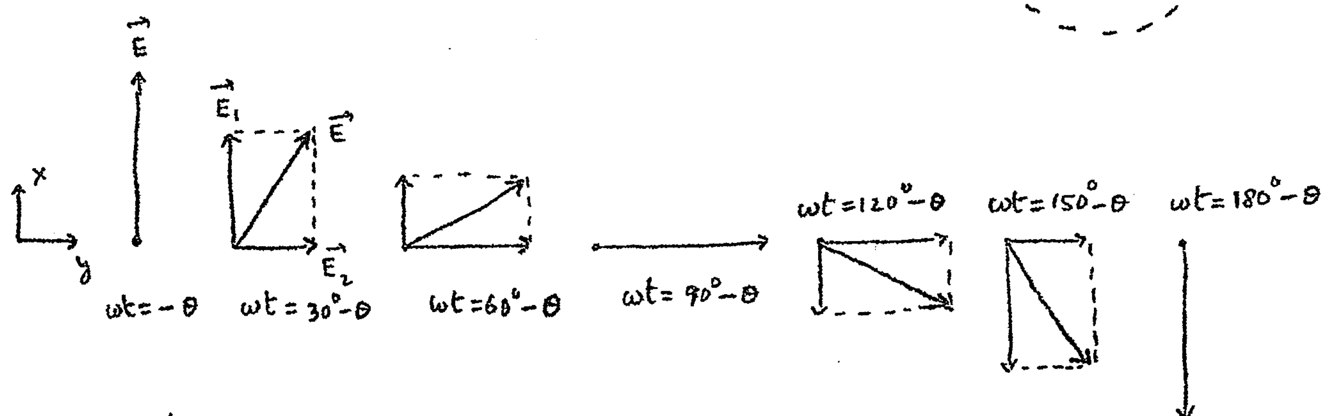
The two fields differ in direction by 90°, differ in phase by 90° but have equal amplitudes.

$$|\vec{E}| = \left[E_0^2 \cos^2(\omega t + \theta) + E_0^2 \sin^2(\omega t + \theta)\right]^{1/2} = E_0$$

Direction of $\vec{E}$: $\alpha = \tan^{-1} \dfrac{E_0 \sin(\omega t + \theta)}{E_0 \cos(\omega t + \theta)} = \omega t + \theta$.



Thus the vector $\vec{E}$ is circularly polarized.



Example:

$$\vec{E}_1 = (3\vec{i}_x - 4\vec{i}_z) \cos \omega t$$
$$\vec{E}_2 = 5\vec{i}_y \sin \omega t$$

$|\vec{E}_1| = |3\vec{i}_x - 4\vec{i}_z| = 5$, $|\vec{E}_2| = 5$

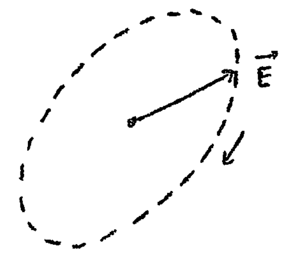
$\vec{E}_1 \cdot \vec{E}_2 = (3\vec{i}_x - 4\vec{i}_z) \cdot 5\vec{i}_y = 0$

$\vec{E}_1$ and $\vec{E}_2$ are out of phase by 90°.

Hence $\vec{E}_1 + \vec{E}_2$ is circularly polarized.

Elliptical Polarization:

This is the general case in which the two vectors differ in magnitude, direction and phase by arbitrary amounts.



Example:

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$= 1 \cos(\omega t + 75^\circ)\,\vec{i}_x + \sqrt{2}\cos(\omega t + 30^\circ)\,\vec{i}_y$$

The two vectors differ in magnitude, differ in phase by 45°, and differ in direction by 90°. Hence $\vec{E}$ is elliptically polarized.

Homework Problems (due 9/6/74)

1. Find $10\cos(\omega t + 30^\circ) + 10\sin\omega t$ by using the phasor technique.

2. Determine the polarization of the sum vector obtained by adding the two following fields.

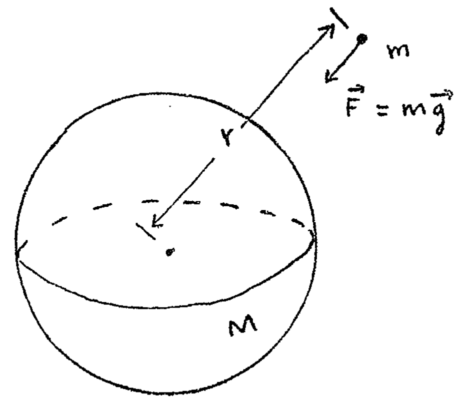
$$\vec{E}_1 = (-\sqrt{3}\,\vec{i}_x + \vec{i}_y)\cos\omega t$$

$$\vec{E}_2 = \left(\frac{1}{2}\,\vec{i}_x + \frac{\sqrt{3}}{2}\,\vec{i}_y - \sqrt{3}\,\vec{i}_z\right)\sin\omega t$$

Gravitational Field:

$$\vec{F} = \frac{mMG}{r^2}$$ directed towards the center of the earth.

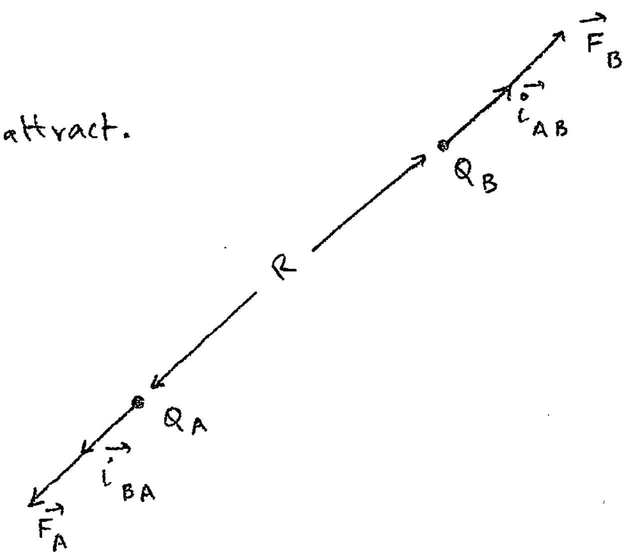
$$\vec{g} = \frac{\vec{F}}{m} = \frac{MG}{r^2}$$ towards the center of the earth.



Gravitational field is a force field associated with the physical property known as "mass."

Likewise, electric field is a force field associated with the physical property known as "charge."

Coulomb's Law:



1. Like charges repel, unlike charges attract.
2. $|\vec{F}| \propto$ product of charges
3. $|\vec{F}| \propto \frac{1}{R^2}$
4. Direction of force along the line joining the charges
5. Force depends upon the medium.

$$\vec{F}_A = \frac{Q_A Q_B}{k R^2} \vec{i}_{BA}$$

$$\vec{F}_B = \frac{Q_B Q_A}{k R^2} \vec{i}_{AB}$$

$$k = \frac{1}{4\pi\epsilon_0}$$

ϵ_0 = permittivity of free space $= \frac{10^{-9}}{36\pi} \frac{C^2}{N-m^2}$ or $\frac{F}{m}$

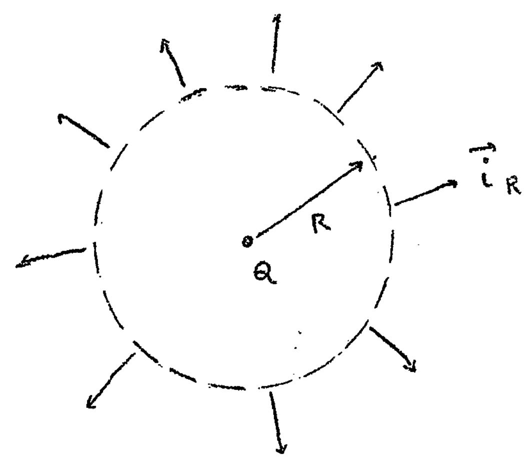
we define $\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q}$

At point A, $\vec{E}_A = \lim_{Q_A \to 0} \frac{\vec{F}_A}{Q_A} = \frac{Q_B}{4\pi\epsilon_0 R^2} \vec{i}_{BA}$

At point B, $\vec{E}_B = \lim_{Q_B \to 0} \frac{\vec{F}_B}{Q_B} = \frac{Q_A}{4\pi\epsilon_0 R^2} \vec{i}_{AB}$

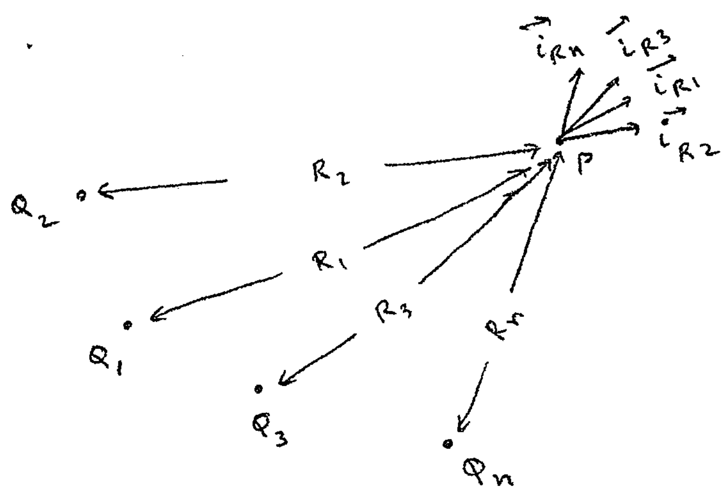
In general, if we place a small stationary test charge q in a region and it experiences a force $\vec{F}$, then we say that there is an electric field $\vec{E}$ in that region given by

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q}$$



Electric field of point charges:

$$\vec{F} = \frac{Q}{4\pi\epsilon_0 R^2} \vec{i}_R$$

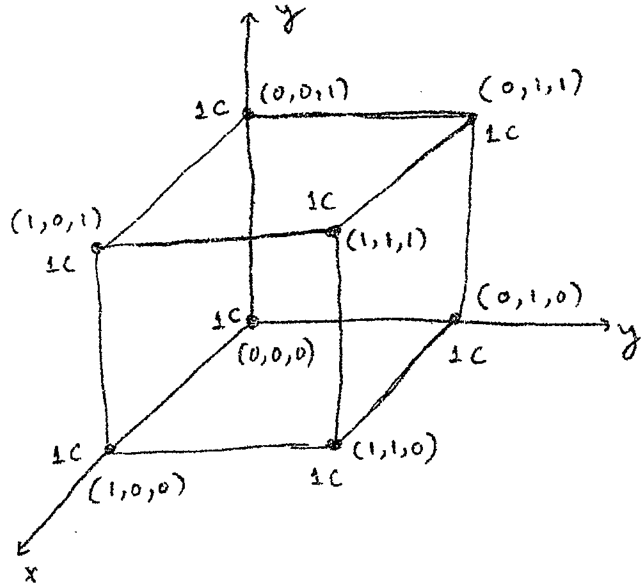


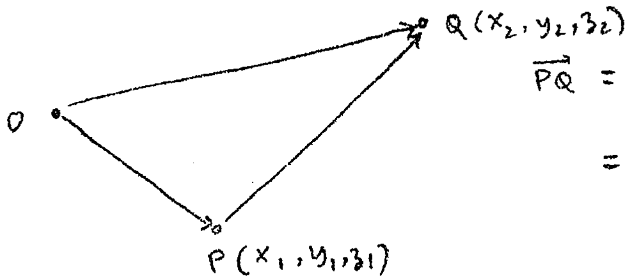
$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 R_1^2} \vec{i}_{R1} + \frac{Q_2}{4\pi\epsilon_0 R_2^2} \vec{i}_{R2} + \cdots + \frac{Q_n}{4\pi\epsilon_0 R_n^2} \vec{i}_{Rn}$$

$$= \sum_{j=1}^{n} \frac{Q_j}{4\pi\epsilon_0 R_j^2} \vec{i}_{R_j}$$

Example:

Find the force acting on each charge.





$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (x_2 \vec{i}_x + y_2 \vec{i}_y + z_2 \vec{i}_z) - (x_1 \vec{i}_x + y_1 \vec{i}_y + z_1 \vec{i}_z)$$
$$= (x_2 - x_1)\vec{i}_x + (y_2 - y_1)\vec{i}_y + (z_2 - z_1)\vec{i}_z$$

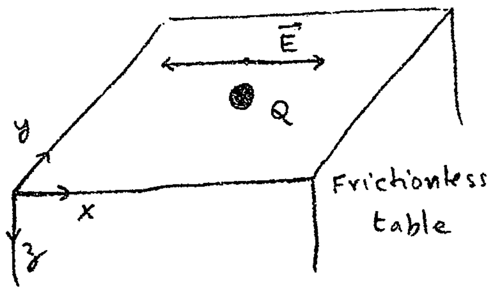
$$\vec{i}_{PQ} = \frac{\overrightarrow{PQ}}{PQ} = \frac{(x_2-x_1)\vec{i}_x + (y_2-y_1)\vec{i}_y + (z_2-z_1)\vec{i}_z}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

Consider the charge at (1,1,1)

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left[\vec{i}_x + \vec{i}_y + \vec{i}_z + \frac{1}{2}\left(\frac{\vec{i}_y + \vec{i}_z}{\sqrt{2}} + \frac{\vec{i}_z + \vec{i}_x}{\sqrt{2}} + \frac{\vec{i}_x + \vec{i}_y}{\sqrt{2}}\right) + \frac{1}{3}\,\frac{\vec{i}_x + \vec{i}_y + \vec{i}_z}{\sqrt{3}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right](\vec{i}_x + \vec{i}_y + \vec{i}_z) = \frac{3.29}{4\pi\epsilon_0} \cdot \frac{\vec{i}_x + \vec{i}_y + \vec{i}_z}{\sqrt{3}}$$

∴ The force experienced by each charge is equal to $\frac{3.29}{4\pi\epsilon_0}$ N and it is directed away from the corner opposite to that charge.



$$\vec{E} = E_0 \cos\omega t \, \vec{i}_x$$

$$\vec{F} = Q\vec{E}$$

$$m\frac{d\vec{v}}{dt} = Q E_0 \cos\omega t \, \vec{i}_x$$

$$\frac{d\vec{v}}{dt} = \frac{Q E_0}{m} \cos\omega t \, \vec{i}_x$$

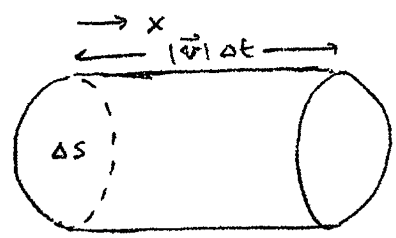
$$\vec{v} = \frac{Q E_0}{m\omega} \sin\omega t \, \vec{i}_x + \overset{=0}{c_1}$$

$$x = -\frac{Q E_0}{m\omega^2} \cos\omega t + c_2$$

Let us now consider a cloud of electrons distributed uniformly with density N. Then each electron experiences a velocity

$$\vec{v} = \frac{e E_0}{m\omega} \sin\omega t \, \vec{i}_x = -\frac{|e| E_0}{m\omega} \sin\omega t \, \vec{i}_x$$

Let us consider an area ΔS normal to the x direction as shown in the figure below



Number of electrons crossing to the left of the area ΔS in a time Δt are the same as those which exist in a column of length $|\vec{v}|\Delta t$ and crosssectional area ΔS to the right of the area under consideration.

$\therefore$ charge ΔQ crossing to the left of the area ΔS in time Δt $\Big\}$ $= (\Delta S)(|\vec{v}|\Delta t)\, Ne$

$= Ne|\vec{v}|\,\Delta S\,\Delta t$

or, Current ΔI flowing through the area ΔS

$$= \frac{|\Delta Q|}{\Delta t} = N\,|e|\,|\vec{v}|\,\Delta S$$

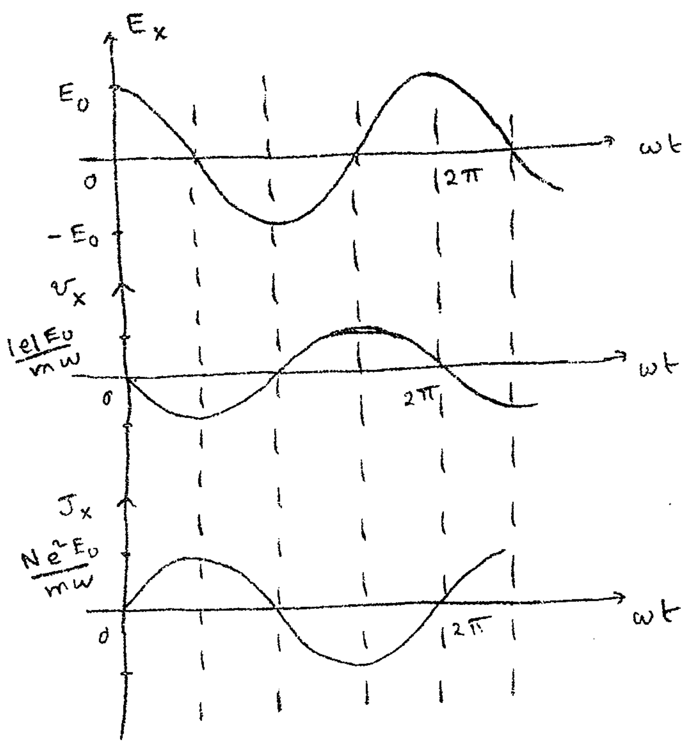
$$= \frac{N\,|e|\,|e|\,E_0}{m\omega} \sin \omega t \cdot \Delta S$$

$$= \frac{N e^2}{m\omega} E_0\, \Delta S \sin \omega t .$$

We define current density as the current per unit area. Current density is a vector directed normal to the area which maximizes the current through it keeping the area size same. Here

$$\text{current density},\ \vec{J} = \frac{\Delta I}{\Delta S}\,\vec{i}_x = \frac{Ne^2}{m\omega} E_0 \sin \omega t\ \vec{i}_x$$

$$\boxed{\vec{J} = N e \vec{v}}$$

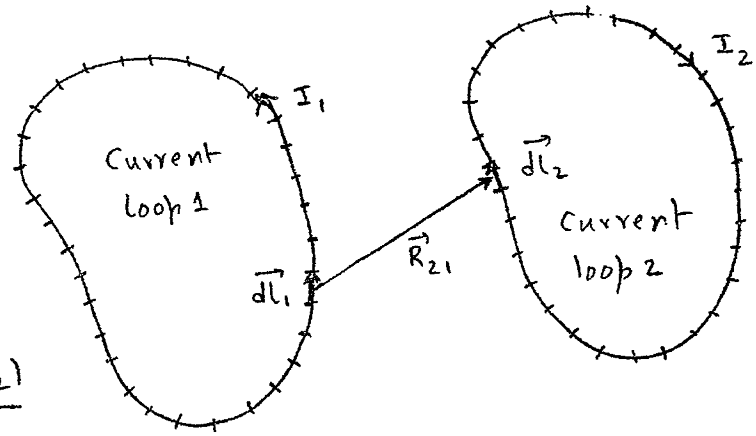


Homework Problem (due 9/9/74)

Three point charges, each of mass m and charge Q, are suspended by strings of length L from a common point. It is found that the common point and the points occupied by the three charges form the corners of an equilateral tetrahedron. Find the relationship between Q, m, L, and the acceleration due to gravity, g.

Ans:
$$\frac{Q^2}{\epsilon_0 L^2 m g} = \frac{4\pi}{\sqrt{6}} .$$

Ampere's law of force between two current loops:



$$\vec{F}_2 = k \oint_{C_1} \oint_{C_2} \frac{I_2 \vec{dl}_2 \times (I_1 \vec{dl}_1 \times \vec{R}_{21})}{R_{21}^3}$$

$$\vec{F}_1 = k \oint_{C_2} \oint_{C_1} \frac{I_1 \vec{dl}_1 \times (I_2 \vec{dl}_2 \times \vec{R}_{12})}{R_{12}^3}$$

$$k = \frac{\mu_0}{4\pi}$$

μ_0 = permeability of free space $= 4\pi \times 10^{-7}$ h/m

Force between current elements:

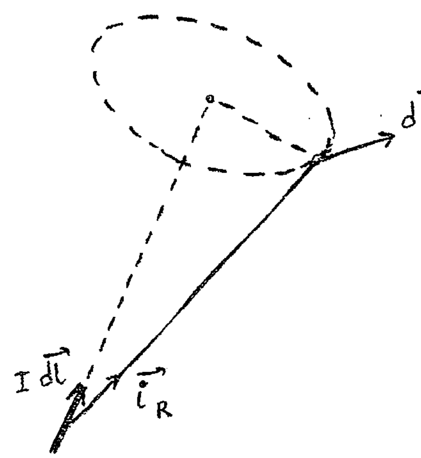
$$d\vec{F}_{21} = \frac{\mu_0}{4\pi} I_2 \vec{dl}_2 \times I_1 \frac{\vec{dl}_1 \times \vec{R}_{21}}{R_{21}^3} = I_2 \vec{dl}_2 \times \underbrace{\left(\frac{\mu_0}{4\pi} \frac{I_1 \vec{dl}_1 \times \vec{R}_{21}}{R_{21}^3} \right)}_{d\vec{B}_{21}}$$

$$d\vec{F}_{12} = I_1 \vec{dl}_1 \times \underbrace{\left(\frac{\mu_0}{4\pi} \frac{I_2 \vec{dl}_2 \times \vec{R}_{12}}{R_{12}^3} \right)}_{d\vec{B}_{12}}$$

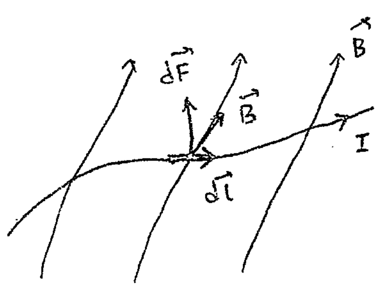
In general,

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, \vec{dl} \times \vec{R}}{R^3} = \frac{\mu_0}{4\pi} \frac{I\, \vec{dl} \times \vec{i}_R}{R^2}}$$

Biot-Savart Law

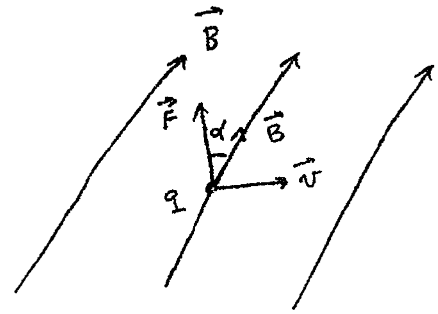


$$d\vec{F} = I\, \vec{dl} \times \vec{B}$$
$$= \frac{dQ}{dt} \vec{dl} \times \vec{B}$$
$$= dQ \frac{\vec{dl}}{dt} \times \vec{B}$$
$$= dQ\, \vec{v} \times \vec{B}$$



Definition of magnetic field in terms of force on a moving test charge:

If a test charge q moving with a velocity $\vec{v}$ experiences a force $\vec{F}$ by virtue of it's movement, then the region is said to be characterized by a magnetic field $\vec{B}$ such that



$$\boxed{\vec{F} = q\vec{v} \times \vec{B}}$$

$$|\vec{F}| = qvB\sin\alpha$$

direction of $\vec{F}$ = perpendicular to both $\vec{v}$ and $\vec{B}$.

$$\vec{B} = \lim_{qv \to 0} \frac{\vec{F}_m \times \vec{i}_m}{qv}$$

where $\vec{F}_m$ is the maximum force experienced by q and $\vec{i}_m$ is the unit vector along the direction of $\vec{v}$ for which this maximum force is observed.

Units of $\vec{B}$ are $\dfrac{\text{newtons}}{\text{Coulomb} \times \text{meter/sec}} = \dfrac{\text{newtons}}{\text{Coulomb}} \times \dfrac{\text{second}}{\text{meter}}$

$$= \frac{\text{newton-meters}}{\text{coulomb}} \times \frac{\text{second}}{(\text{meter})^2} = \frac{\text{Volt-seconds}}{(\text{meter})^2}$$

$$= \frac{\text{webers}}{(\text{meter})^2} = \text{Wb/m}^2$$

$\vec{B}$ is known as the magnetic flux density vector.

<u>Lorentz Force Equation:</u>

If both $\vec{E}$ and $\vec{B}$ exist in a region, then

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = q(\vec{E} + \vec{v} \times \vec{B})}$$

Example 1. The forces experienced by a test charge for three different velocities are given by

$\vec{v}_1 = \vec{i}_x$ $\qquad \vec{F}_1 = q\,\vec{i}_x$

$\vec{v}_2 = \vec{i}_y$ $\qquad \vec{F}_2 = q\,(2\vec{i}_x + \vec{i}_y)$

$\vec{v}_3 = \vec{i}_z$ $\qquad \vec{F}_3 = q\,(\vec{i}_x + \vec{i}_y)$

Find $\vec{E}$ and $\vec{B}$.

From Lorentz force equation, we have

$$q\,\vec{i}_x = q\vec{E} + q\,\vec{i}_x \times \vec{B} \qquad \text{— (1)}$$

$$q(2\vec{i}_x + \vec{i}_y) = q\vec{E} + q\,\vec{i}_y \times \vec{B} \qquad \text{— (2)}$$

$$q(\vec{i}_x + \vec{i}_y) = q\vec{E} + q\,\vec{i}_z \times \vec{B} \qquad \text{— (3)}$$

$$(2) - (1) \longrightarrow (\vec{i}_y - \vec{i}_x) \times \vec{B} = \vec{i}_x + \vec{i}_y \qquad \text{— (4)}$$

$$(2) - (3) \longrightarrow (\vec{i}_y - \vec{i}_z) \times \vec{B} = \vec{i}_x \qquad \text{— (5)}$$

Applying the vector identity

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = (\vec{A} \times \vec{B} \cdot \vec{D})\,\vec{C} - (\vec{A} \times \vec{B} \cdot \vec{C})\,\vec{D}$$

to (4) and (5), we get

$$[(\vec{i}_y - \vec{i}_x) \times \vec{B}] \times [(\vec{i}_y - \vec{i}_z) \times \vec{B}] = (\vec{i}_x + \vec{i}_y) \cdot \vec{i}_x$$

$$[(\vec{i}_y - \vec{i}_x) \times \vec{B} \cdot \vec{B}]\,(\vec{i}_y - \vec{i}_z) - [(\vec{i}_y - \vec{i}_x) \times \vec{B} \cdot (\vec{i}_y - \vec{i}_z)]\,\vec{B} = -\vec{i}_z$$

$$0 - [(\vec{i}_x + \vec{i}_y) \cdot (\vec{i}_y - \vec{i}_z)]\,\vec{B} = -\vec{i}_z$$

$$\boxed{\vec{B} = \vec{i}_z}, \qquad \vec{E} = \vec{i}_x - \vec{i}_x \times \vec{B} = \vec{i}_x - \vec{i}_x \times \vec{i}_z = \vec{i}_x + \vec{i}_y$$

$$\boxed{\vec{E} = \vec{i}_x + \vec{i}_y}$$

Verification:

$$(\vec{i}_x + \vec{i}_y) + \vec{i}_x \times \vec{i}_z = \vec{i}_x$$

$$(\vec{i}_x + \vec{i}_y) + \vec{i}_y \times \vec{i}_z = 2\vec{i}_x + \vec{i}_y$$

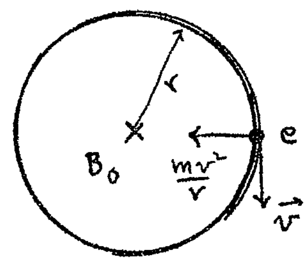
$$(\vec{i}_x + \vec{i}_y) + \vec{i}_z \times \vec{i}_z = \vec{i}_x + \vec{i}_y$$

Example 2. Let us consider a steady, uniform magnetic field B_0 directed along the z axis and consider the motion of an electron moving in a circular orbit in the plane normal to the z axis.

The motion of the electron is obviously governed by the condition that the centripetal force keeping the electron in the circular orbit must equal the magnetic force exerted on the electron. Thus

$$\frac{mv^2}{r} = |e|\, v\, B_0$$

or $$r = \frac{mv}{|e|\, B_0}$$



The number of orbits made by the electron in one second $= \dfrac{1}{2\pi r/v} = \dfrac{|e|\, B_0}{2\pi m}$.

This quantity is known as the gyrofrequency, f_H.

For the earth's magnetic field at ionospheric heights, $f_H \approx 1.4$ MHz

Example 3. Let us consider the motion of an electron in a region of crossed electric and magnetic fields

$$\vec{E} = E_0 \cos\omega t\, \vec{i}_x \quad \text{and} \quad \vec{B} = B_0 \vec{i}_z .$$

From Lorentz force equation, we have

$$m \frac{d\vec{v}}{dt} = e\left(E_0 \cos\omega t\, \vec{i}_x + \vec{v} \times B_0 \vec{i}_z \right) \quad \text{— (1)}$$

Equating the like components on either side of (1), we get

$$m \frac{dv_x}{dt} = e E_0 \cos\omega t + e v_y B_0 \quad \text{— (2)}$$

$$m \frac{dv_y}{dt} = - e v_x B_0 \quad \text{— (3)}$$

Eliminating v_y from (2) and (3), we obtain

$$m \frac{d^2 v_x}{dt^2} = -eE_0 \omega \sin \omega t + eB_0 \frac{dv_y}{dt}$$

$$= -eE_0 \omega \sin \omega t - \frac{e^2 B_0^2}{m} v_x$$

or $$\frac{d^2 v_x}{dt^2} + \frac{e^2 B_0^2}{m^2} v_x = -\frac{eE_0}{m} \omega \sin \omega t \qquad — (4)$$

or $$\frac{d^2 v_x}{dt^2} + \omega_H^2 v_x = -\frac{eE_0}{m} \omega \sin \omega t \quad \text{where } \omega_H = \frac{|e| B_0}{m}. \qquad — (5)$$

Assuming steady state solution for $v_x = A \cos \omega t + B \sin \omega t$ and substituting in (5), we have

$$-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t + \omega_H^2 (A \cos \omega t + B \sin \omega t) = -\frac{eE_0}{m} \omega \sin \omega t$$

or $$-A\omega^2 + \omega_H^2 A = 0 \qquad \text{or} \quad A = 0$$

$$-B\omega^2 + \omega_H^2 B = -\frac{eE_0 \omega}{m}$$

or $$B = \frac{eE_0 \omega}{m(\omega^2 - \omega_H^2)}$$

Thus $$v_x = \frac{e\omega}{m(\omega^2 - \omega_H^2)} E_0 \sin \omega t$$

$$v_y = -\frac{eB_0}{m} \int v_x \, dt = -\frac{e\omega_H}{m(\omega^2 - \omega_H^2)} E_0 \cos \omega t.$$

$$\vec{v} = v_x \vec{i}_x + v_y \vec{i}_y = \frac{eE_0}{m(\omega^2 - \omega_H^2)} \left[\omega \sin \omega t \, \vec{i}_x - \omega_H \cos \omega t \, \vec{i}_y \right]$$

$$\vec{J} = Ne\vec{v} = \frac{Ne^2 E_0}{m(\omega^2 - \omega_H^2)} \left[\omega \sin \omega t \, \vec{i}_x - \omega_H \cos \omega t \, \vec{i}_y \right]$$

Note that $\vec{J}$ is elliptically polarized.

Homework Problem (due 9/11/74)

The forces experienced by a test charge q at a point in a region of electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively, are given as follows for three different velocities:

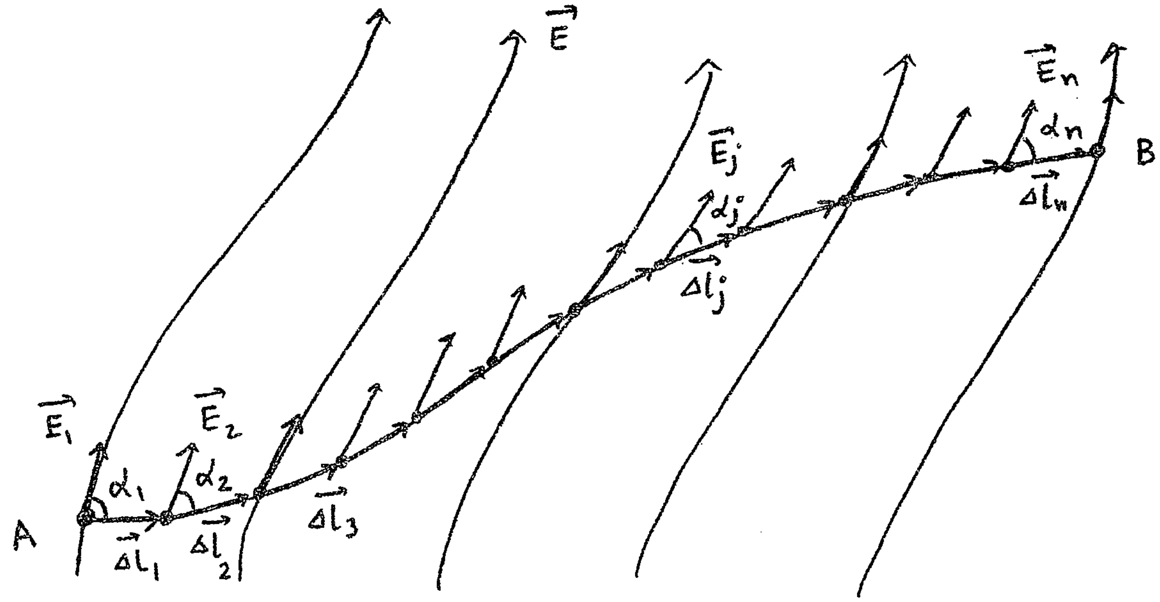
$$\vec{v}_1 = \vec{i}_y - \vec{i}_x \quad , \quad \vec{F}_1 = q\,(2\vec{i}_x + 2\vec{i}_y)$$

$$\vec{v}_2 = \vec{i}_y - \vec{i}_z \quad , \quad \vec{F}_2 = q(2\vec{i}_x + \vec{i}_y)$$

$$\vec{v}_3 = \vec{i}_x - \vec{i}_y + \vec{i}_z \quad , \quad \vec{F}_3 = 0$$

Find $\vec{E}$ and $\vec{B}$.

LINE INTEGRAL



Work done in carrying a test charge from A to B:

$$W_A^B = q\,E_1 \cos\alpha_1\,\Delta l_1 + q E_2 \cos\alpha_2\;\Delta l_2 + \cdots + q\,E_n \cos\alpha_n\,\Delta l_n$$

$$= q \sum_{j=1}^{n} E_j \cos\alpha_j \;\Delta l_j$$

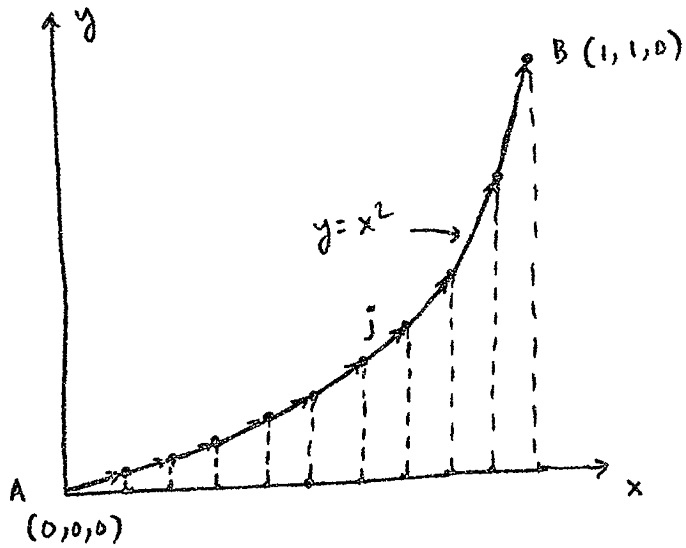
$$= q \sum_{j=1}^{n} \vec{E}_j \cdot \vec{\Delta l}_j$$

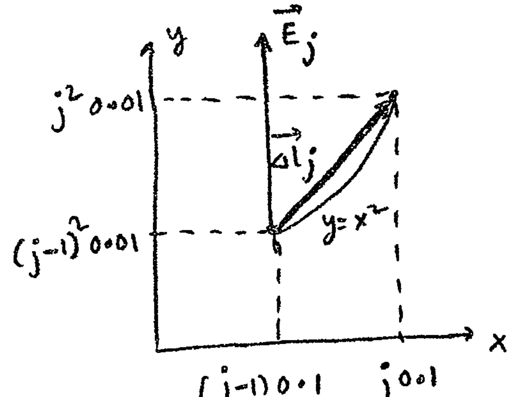
In the limit that $n \to \infty$,

$$W_A^B = \lim_{n \to \infty} q \sum_{j=1}^{n} \vec{E}_j \cdot \vec{\Delta l}_j = q \int_A^B \vec{E} \cdot d\vec{l}$$

$\int_A^B \vec{E} \cdot d\vec{l}$ is known as the line integral of $\vec{E}$ from A to B. It is the work done per unit charge by the electric field in carrying a test charge from A to B. It is known as the voltage between A and B.

EXAMPLE: Consider $\vec{E} = x\,\vec{i}_y$ and the path $y = x^2$, $z = 0$ from the point $A(0,0,0)$ to the point $B(1,1,0)$.





We wish to find the line integral of $\vec{E}$ from A to B.

a) Approximate value $= \sum_{j=1}^{10} \vec{E}_j \cdot \vec{\Delta l}_j$

$$\vec{E}_j = (j-1)\,0.1\,\vec{i}_y$$

$$\vec{\Delta l}_j = 0.1\,\vec{i}_x + [\,j^2 - (j-1)^2\,]\,0.01\,\vec{i}_y = 0.1\,\vec{i}_x + (2j-1)\,0.01\,\vec{i}_y$$

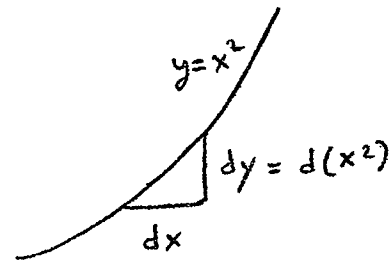
$$\vec{E}_j \cdot \vec{\Delta l}_j = (j-1)(2j-1)\,0.001$$

$$\sum_{j=1}^{10} \vec{E}_j \cdot \vec{\Delta l}_j = \sum_{j=1}^{10} (j-1)(2j-1)\,0.001$$

$$= 0.001\,(0 + 3 + 10 + 21 + 36 + 55 + 78 + 105 + 136 + 171)$$

$$= 0.001 \times 615 = 0.615 \text{ N-m/C or } 0.615 \text{ V}.$$

b) Exact value $= \int_A^B \vec{E} \cdot \vec{dl}$



$$\vec{E} = x\,\vec{i}_y$$

$$\vec{dl} = dx\,\vec{i}_x + dy\,\vec{i}_y + dz\,\vec{i}_z$$

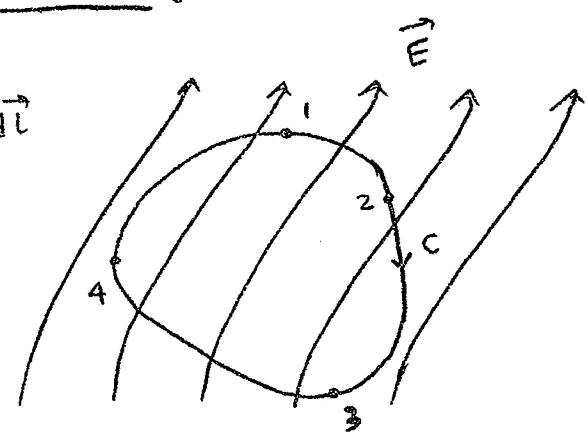
$$= dx\,\vec{i}_x + d(x^2)\,\vec{i}_y + 0$$

$$= dx\,\vec{i}_x + 2x\,dx\,\vec{i}_y$$

$$\vec{E} \cdot \vec{dl} = (x\,\vec{i}_y) \cdot (dx\,\vec{i}_x + 2x\,dx\,\vec{i}_y) = 2x^2\,dx$$

$$\int_A^B \vec{E} \cdot \vec{dl} = \int_{(0,0,0)}^{(1,1,0)} \vec{E} \cdot \vec{dl} = \int_{x=0}^{1} 2x^2\,dx = \frac{2}{3}\ \text{V}.$$

LINE INTEGRAL AROUND CLOSED PATH :



$$\oint_C \vec{E} \cdot \vec{dl} = \int_{123} \vec{E} \cdot \vec{dl} + \int_{341} \vec{E} \cdot \vec{dl}$$

$$= \int_1^2 \vec{E} \cdot \vec{dl} + \int_2^3 \vec{E} \cdot \vec{dl}$$

$$+ \int_3^4 \vec{E} \cdot \vec{dl} + \int_4^1 \vec{E} \cdot \vec{dl}$$

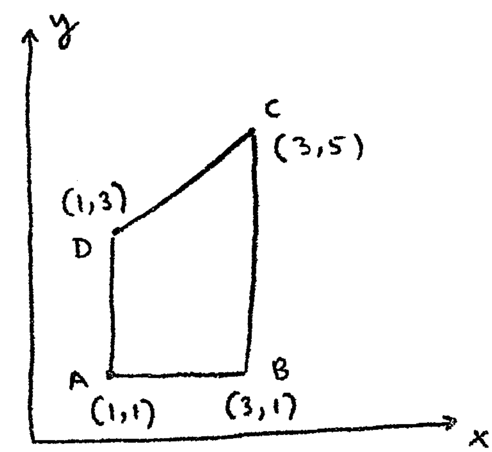
$\oint_C \vec{E} \cdot \vec{dl}$ is the work done per unit charge by the field in moving a test charge around the closed path c. It has several names:

Circulation of $\vec{E}$ around c

Electromotive force (emf)

Voltage

EXAMPLE: $\vec{E} = x\,\vec{i_y}$



From A to B,

$y = 1,\ dy = 0,\ \vec{dl} = dx\,\vec{i_x}$

$\vec{E}\cdot\vec{dl} = x\,\vec{i_y}\cdot dx\,\vec{i_x} = 0$

$\int_A^B \vec{E}\cdot\vec{dl} = 0$ (Field $\perp^r$ to the path)

From B to C,

$x = 3,\ dx = 0,\ \vec{dl} = dy\,\vec{i_y}$

$\vec{E}\cdot\vec{dl} = 3\,\vec{i_y}\cdot dy\,\vec{i_y} = 3\,dy$

$\int_B^C \vec{E}\cdot\vec{dl} = \int_1^5 3\,dy = 12$

From C to D,

$y = 2 + x,\ dy = dx,\ \vec{dl} = dx\,\vec{i_x} + dx\,\vec{i_y}$

$\vec{E}\cdot\vec{dl} = (x\,\vec{i_y})\cdot(dx\,\vec{i_x} + dx\,\vec{i_y}) = x\,dx$

$\int_C^D \vec{E}\cdot\vec{dl} = \int_3^1 x\,dx = -4$

From D to A,

$x = 1,\ dx = 0,\ \vec{dl} = dy\,\vec{i_y}$

$\vec{E}\cdot\vec{dl} = 1\,\vec{i_y}\cdot dy\,\vec{i_y} = dy$

$\int_D^A \vec{E}\cdot\vec{dl} = \int_3^1 dy = -2.$

$$\oint_{ABCDA} \vec{E}\cdot\vec{dl} = \int_A^B \vec{E}\cdot\vec{dl} + \int_B^C \vec{E}\cdot\vec{dl} + \int_C^D \vec{E}\cdot\vec{dl} + \int_D^A \vec{E}\cdot\vec{dl}$$

$$= 0 + 12 + (-4) + (-2) = 6.$$

Homework Problem (due 9/13/74)

Given a force field $\vec{F} = x^2 \vec{i}_y$,

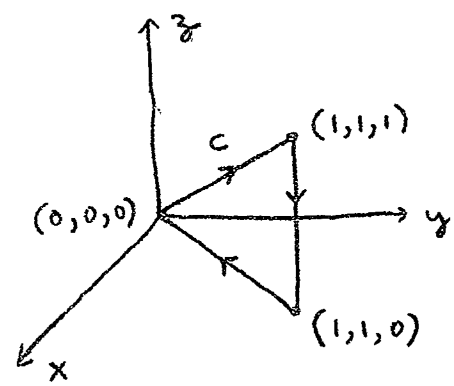
(a) evaluate the approximate value of the line integral of $\vec{F}$ from the origin (0,0,0) to the point (1,3,0) along a straight line path, by dividing the path into 10 equal segments.

(b) evaluate the exact value of the line integral of $\vec{F}$ for the path specified in (a).

Additional Problems:

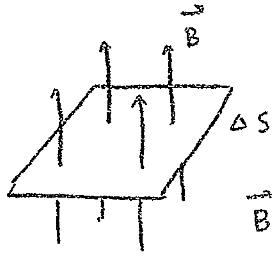
1. Given $\vec{E} = y\vec{i}_x + x\vec{i}_y$, find $\int_{(0,0,0)}^{(1,1,0)} \vec{E} \cdot d\vec{l}$ along the following paths: (Answer is 1 for all paths. Can you explain?)
   (a) straight line path $y = x, z = 0$
   (b) straight line path from (0,0,0) to (1,0,0) and then straight line path from (1,0,0) to (1,1,0)
   (c) Any path of your choice.

2. Given $\vec{E} = xy\,\vec{i}_x + yz\,\vec{i}_y + zx\,\vec{i}_z$, find $\oint_C \vec{E} \cdot d\vec{l}$ where C is the closed path shown in the accompanying figure.



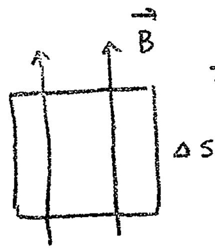
NEXT: SURFACE INTEGRAL

SURFACE INTEGRAL



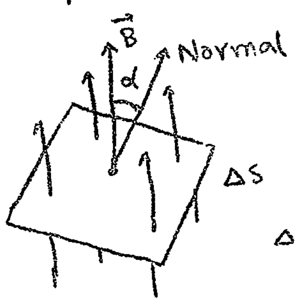
$\vec{B} \perp^{r} \Delta S$

$\Delta\psi = B\ \Delta S$



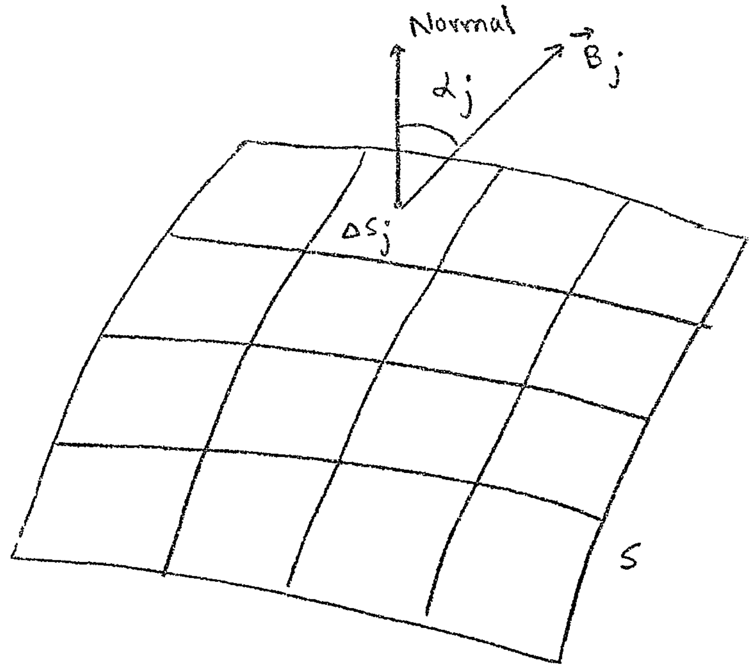
$\vec{B} \parallel^{l} \Delta S$

$\Delta\psi = 0$



$\angle^{\vec{B}}_{\text{Normal}} = \alpha$

$\Delta\psi = B \cos\alpha\ \Delta S$



Magnetic flux crossing the surface S is given by

$$[\psi]_S = \Delta\psi_1 + \Delta\psi_2 + \Delta\psi_3 + \cdots + \Delta\psi_n$$

$$= B_1 \cos\alpha_1\ \Delta S_1 + B_2 \cos\alpha_2\ \Delta S_2 + \cdots + B_n \cos\alpha_n\ \Delta S_n$$

$$= \sum_{j=1}^{n} B_j \cos\alpha_j\ \Delta S_j = \sum_{j=1}^{n} \vec{B}_j \cdot \Delta S_j\ \vec{i}_{nj}$$

$$= \sum_{j=1}^{n} \vec{B}_j \cdot \vec{\Delta S}_j$$

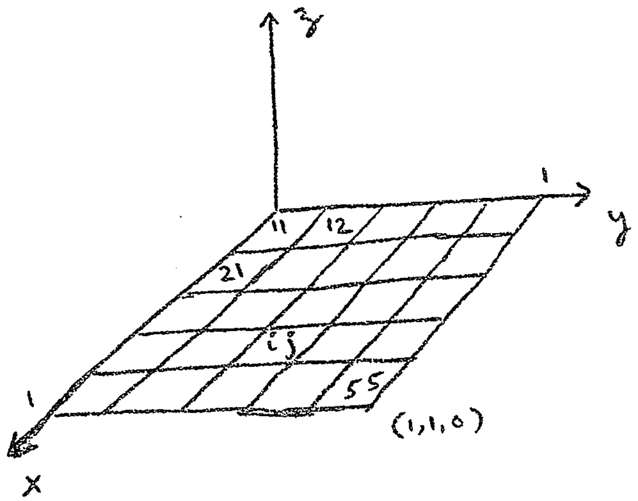
where $\vec{\Delta S}_j = \Delta S_j\ \vec{i}_{nj}$ and $\vec{i}_{nj}$ is unit vector normal to ΔS_j.

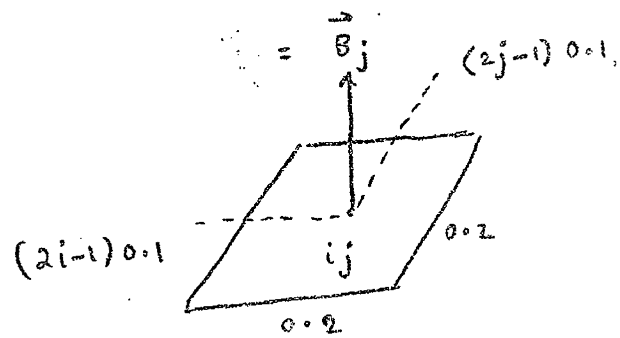
In the limit that $n \to \infty$, we get the exact value of the magnetic flux,

$$[\psi]_S = \lim_{n\to\infty} \sum_{j=1}^{n} \vec{B}_j \cdot \vec{\Delta S}_j = \int_S \vec{B} \cdot d\vec{S}$$

known as the surface integral of $\vec{B}$ over S.

EXAMPLE: Consider $\vec{B} = 3xy^2 \vec{i}_z$ wb/m² and the portion of the xy plane lying between $x = 0$, $x = 1$, $y = 0$, and $y = 1$.





Approximate value of ψ:

$$\vec{B}_{ij} = 3(2i-1)(2j-1)^2\, 0.01\, \vec{i}_z$$

$$\vec{\Delta s}_{ij} = 0.04\, \vec{i}_z$$

$$\psi = \sum_{i=1}^{5} \sum_{j=1}^{5} \left[3(2i-1)(2j-1)^2\, 0.01\, \vec{i}_z\right] \cdot \left[0.04\, \vec{i}_z\right]$$

$$= 0.00012 \sum_{i=1}^{5} \sum_{j=1}^{5} (2i-1)(2j-1)^2$$

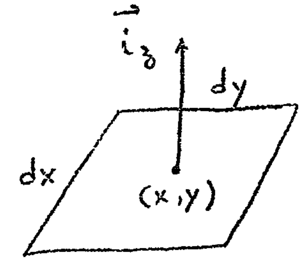
$$= 0.00012 \sum_{i=1}^{5} (2i-1) \sum_{j=1}^{5} (2j-1)^2$$

$$= 0.00012\,(1+3+5+7+9)(1+9+25+49+81)$$

$$= 0.00012 \times 25 \times 165$$

$$= 0.495 \text{ wb.}$$

Exact Value of Ψ :



$$\vec{B} = 3xy^2 \, \vec{i}_z$$

$$\vec{ds} = dx \, dy \, \vec{i}_z$$

$$\vec{B} \cdot \vec{ds} = 3xy^2 \vec{i}_z \cdot dx \, dy \, \vec{i}_z$$

$$= 3xy^2 \, dx \, dy$$

$$\Psi = \int_s \vec{B} \cdot \vec{ds} = \int_{x=0}^{1} \int_{y=0}^{1} 3xy^2 \, dx \, dy$$

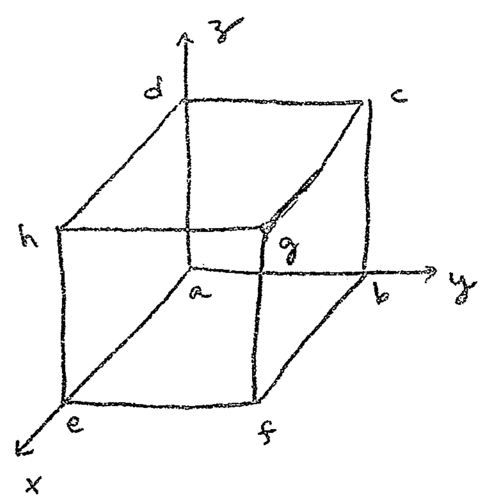
$$= 3 \int_{x=0}^{1} x \left[\frac{y^3}{3} \right]_0^1 dx = \int_0^1 x \, dx = 0.5 \text{ wb}$$

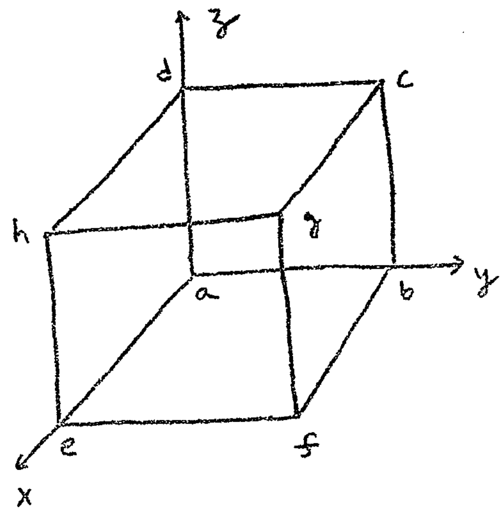
If the surface integral is evaluated over a closed surface, then it is written as $\oint_s \vec{B} \cdot \vec{ds}$

EXAMPLE: Consider $\vec{B} = (x+2)\vec{i}_x + (1-3y)\vec{i}_y + 2z\,\vec{i}_z$ and the cubical box bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

We wish to find $\oint_S \vec{B} \cdot \vec{ds}$ where S is the surface of the box.

To do this, we find $\int \vec{B} \cdot \vec{ds}$ for each of the six surfaces and add up the results. For each surface, we choose the normal vector pointing out of the box.





<u>surface abcd</u> $(x=0)$

$$\vec{B} = \left[(x+2)\vec{i}_x + (1-3y)\vec{i}_y + 2z\vec{i}_z\right]_{x=0}$$
$$= 2\vec{i}_x + (1-3y)\vec{i}_y + 2z\vec{i}_z$$
$$\vec{ds} = dy\,dz\,(-\vec{i}_x) = -dy\,dz\,\vec{i}_x$$
$$\vec{B}\cdot\vec{ds} = -2\,dy\,dz$$
$$\int_{abcd}\vec{B}\cdot\vec{ds} = \int_{z=0}^{1}\int_{y=0}^{1}(-2)\,dy\,dz = -2$$

<u>surface efgh</u> $(x=1)$

$$\vec{B} = 3\vec{i}_x + (1-3y)\vec{i}_y + 2z\vec{i}_z\,,\quad \vec{ds} = dy\,dz\,\vec{i}_x\,,\quad \vec{B}\cdot\vec{ds} = 3\,dy\,dz$$
$$\int_{efgh}\vec{B}\cdot\vec{ds} = \int_{z=0}^{1}\int_{y=0}^{1} 3\,dy\,dz = 3$$

<u>surface aehd</u> $(y=0)$

$$\vec{B} = (x+2)\vec{i}_x + 1\,\vec{i}_y + 2z\vec{i}_z\,,\quad \vec{ds} = -dz\,dx\,\vec{i}_y\,,\quad \vec{B}\cdot\vec{ds} = -dz\,dx$$
$$\int_{aehd}\vec{B}\cdot\vec{ds} = \int_{x=0}^{1}\int_{z=0}^{1}(-1)\,dz\,dx = -1$$

<u>surface bfge</u> $(y=1)$

$$\vec{B} = (x+2)\vec{i}_x - 2\vec{i}_y + 2z\vec{i}_z\,,\quad \vec{ds} = dz\,dx\,\vec{i}_y\,,\quad \vec{B}\cdot\vec{ds} = -2\,dz\,dx$$
$$\int_{bfge}\vec{B}\cdot\vec{ds} = \int_{x=0}^{1}\int_{z=0}^{1}(-2)\,dz\,dx = -2$$

<u>surface aefb</u> $(z=0)$

$$\vec{B} = (x+2)\vec{i}_x + (1-3y)\vec{i}_y + 0\,\vec{i}_z\,,\quad \vec{ds} = -dx\,dy\,\vec{i}_z\,,\quad \vec{B}\cdot\vec{ds} = 0$$
$$\int_{aefb}\vec{B}\cdot\vec{ds} = 0$$

Surface dcgh ($z=1$)

$\vec{B} = (x+2)\vec{i}_x + (1-3y)\vec{i}_y + z\vec{i}_z$, $\vec{ds} = dx\,dy\,\vec{i}_z$, $\vec{B}\cdot\vec{ds} = z\,dx\,dy$

$$\int_{dcgh} \vec{B}\cdot\vec{ds} = \int_{y=0}^{1}\int_{x=0}^{1} z\,dx\,dy = z$$

Thus $\oint_S \vec{B}\cdot\vec{ds} = (-2) + 3 + (-1) + (-2) + 0 + 2 = 0.$

Homework Problem (due 9/16/74)

The current density vector $\vec{J}$ is given by

$$\vec{J} = 3x\,\vec{i}_x + (y-3)\,\vec{i}_y + (2+z)\,\vec{i}_z \quad amp/m^2$$

Find $\oint_S \vec{J}\cdot\vec{ds}$, that is, the current flowing out across the surface S of the rectangular box bounded by the planes

$x=0,\ x=1$

$y=0,\ y=2$

$z=0,\ z=3$

Ans: 30 amp.

FARADAY'S LAW:

When a loop of wire is placed in a time varying magnetic field, an electromotive force (emf) or voltage is induced in the loop.

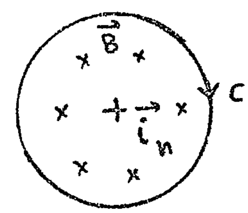
The magnitude of the emf is equal to the magnitude of the time rate of change of the magnetic flux enclosed by the loop.

$$\left| \oint_c \vec{E} \cdot d\vec{l} \right| = \left| \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} \right|$$

where S is any surface bounded by the closed path c. To complete the statement of Faraday's law, we have to consider the polarities of the induced emf and the magnetic flux. For example, if the loop is a circular loop in the plane of the paper, we have four different cases:

$$[emf]_{cw} = \frac{d}{dt}[\Psi]_{\text{into the paper}}$$

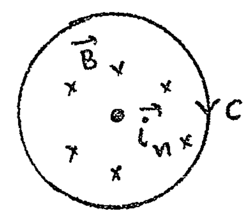
clockwise c $\qquad \vec{i}_n$ into the paper



(a)

$$[emf]_{cw} = \frac{d}{dt}[\Psi]_{\text{out of the paper}}$$

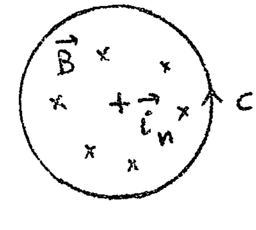
clockwise c $\qquad \vec{i}_n$ out of the paper



(b)

$$[emf]_{ccw} = \frac{d}{dt}[\Psi]_{\text{into the paper}}$$

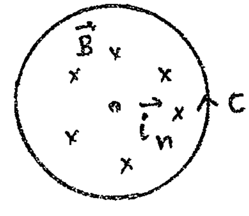
counterclockwise c $\qquad \vec{i}_n$ into the paper



(c) Same as (b)

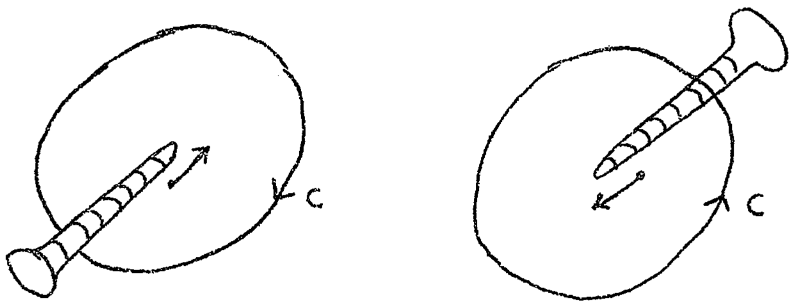
$$[emf]_{ccw} = \frac{d}{dt}[\Psi]_{\text{out of the paper}}$$

Counterclockwise c $\qquad \vec{i}_n$ out of the paper



(d) Same as (a)

According to Faraday's finding, (b) [or (c)] is the correct choice. However, electromagnetic field laws are conventionally formulated in accordance with the right hand screw rule. In other words, the normal vector to the surface is directed towards the advancing direction of a right hand screw when it is turned in the sense in which the loop is traversed. This rule is consistent with case (a) [or (d)].



Hence, to formulate Faraday's finding consistent with the right hand screw rule, we make use of case (a) but include a negative sign on the right side. This gives

$$[\text{emf}]_{cw} = -\frac{d}{dt}[\Psi]_{\text{into the paper}}$$

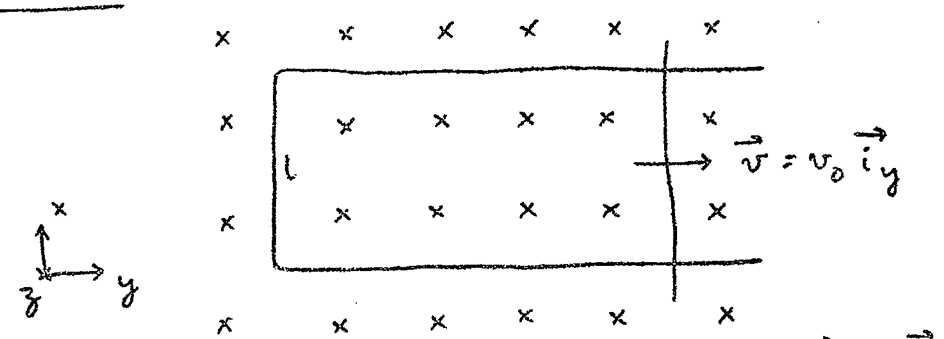
or,

$$[\text{emf}]_{ccw} = -\frac{d}{dt}[\Psi]_{\text{out of the paper}}$$

Thus, we have

$$\boxed{\oint_C \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}}$$

EXAMPLE:



Uniform magnetic field $\vec{B} = B_0 \vec{i}_z$

$$[\psi]_{\text{into the paper}} = \int_S B_0 \vec{i}_z \cdot dS\, \vec{i}_z = B_0 \int_S dS$$

$$= B_0 \text{ (area of the loop) } = B_0\, l\, (y_0 + v_0 t)$$

where y_0 is the position of the moving bar at $t = 0$.

From Faraday's law,

$$[\text{emf induced}]_{\text{clockwise}} = -\frac{d}{dt}[\psi]_{\text{into the paper}}$$

$$= -\frac{d}{dt}[B_0 l (y_0 + v_0 t)] = -B_0 l v_0 .$$

If the bar is moving to the right, the emf produces a current in the counterclockwise sense. This current in turn produces a magnetic field which is directed out of the paper inside the loop, thus opposing the original magnetic field.

<u>EXAMPLE:</u> Consider a time varying magnetic field

$$\vec{B} = B_0 \cos \omega t \ \vec{i}_y$$

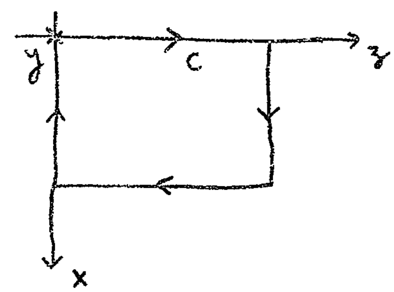
and a fixed rectangular loop as shown in the accompanying figure.

$$[\psi]_{\text{into the paper}} = \int_S \vec{B} \cdot ds \, \vec{i}_y$$

$$= \int_{z=0}^{b} \int_{x=0}^{a} B_0 \cos \omega t \ \vec{i}_y \cdot dx \, dz \, \vec{i}_y$$

$$= B_0 \cos \omega t \int_{z=0}^{b} \int_{x=0}^{a} dx \, dz$$

$$= ab \, B_0 \cos \omega t$$

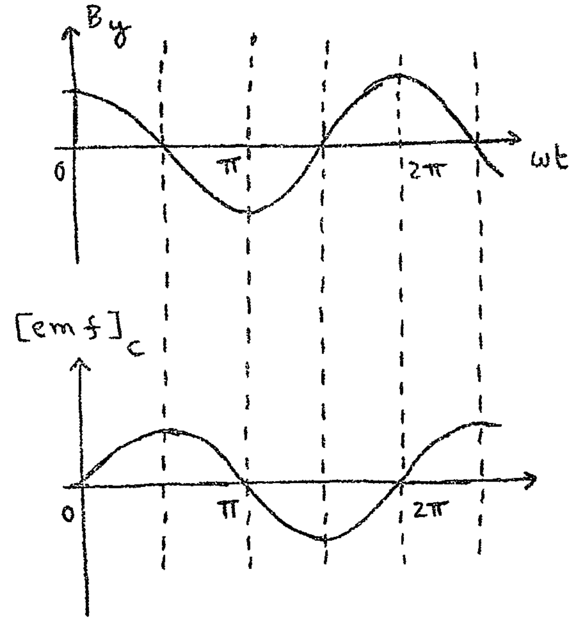


$$[\text{emf induced}]_c = -\frac{d}{dt} [\psi]_{\text{into the paper}}$$

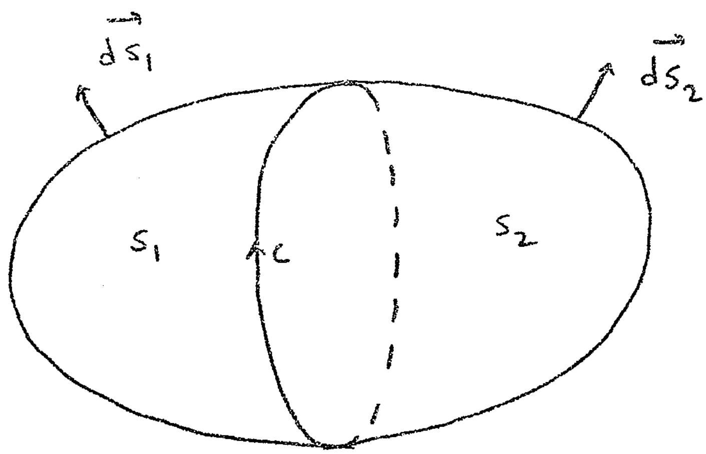
$$= -\frac{d}{dt} [ab \, B_0 \cos \omega t] = ab \, B_0 \, \omega \sin \omega t .$$

$$\oint_c \vec{E} \cdot d\vec{l} = ab B_0 \omega \sin \omega t .$$

Note that once again the emf is induced such that it opposes the change in the magnetic flux.



Let us now consider a closed path C and two surfaces S_1 and S_2 both of which are bounded by C as shown in the figure. Then



$$\oint_C \vec{E} \cdot \vec{dl} = -\frac{d}{dt}\int_{S_1} \vec{B} \cdot \vec{dS_1}$$

$$\oint_C \vec{E} \cdot \vec{dl} = \frac{d}{dt}\int_{S_2} \vec{B} \cdot \vec{dS_2}$$

$$-\frac{d}{dt}\int_{S_1} \vec{B} \cdot \vec{dS_1} = \frac{d}{dt}\int_{S_2} \vec{B} \cdot \vec{dS_2}$$

$$\frac{d}{dt}\int_{S_1} \vec{B} \cdot \vec{dS_1} + \frac{d}{dt}\int_{S_2} \vec{B} \cdot \vec{dS_2} = \frac{d}{dt}\oint_{S_1+S_2} \vec{B} \cdot \vec{dS} = 0$$

$$\oint_{S_1+S_2} \vec{B} \cdot \vec{dS} = \text{constant with time}$$

Since there is no experimental evidence of the above, it follows that

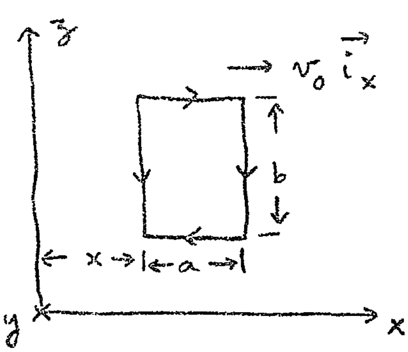
$$\boxed{\oint_S \vec{B} \cdot \vec{dS} = 0}$$ Gauss' law for the magnetic field.

Homework Problem (due 9/18/74)

A magnetic field is given in the xz plane by

$$\vec{B} = \frac{B_0}{|x|} \cos \omega t \, \vec{i}_y$$

where B_0 and ω are constants. A rigid rectangular loop is situated in the xz plane and parallel to the z axis as shown in the figure. If the loop is moving in that plane with a velocity $\vec{v} = v_0 \vec{i}_x$ where v_0 is a constant, find the induced emf around the loop.

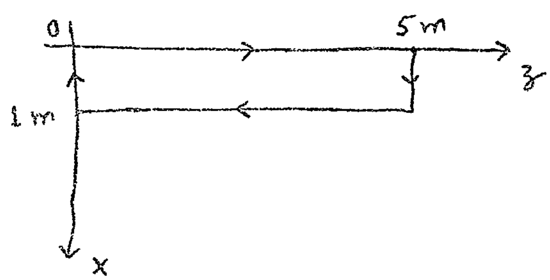


Additional Problem

A time varying magnetic field is given by

$$\vec{B} = B_0 \cos 3\pi \times 10^7 t \cos 0.1\pi z \, \vec{i}_y$$

where B_0 is a constant. Find the induced emf around the rectangular loop in the xz plane shown in the accompanying figure.



AMPERE'S CIRCUITAL LAW:

$$\oint_C \frac{\vec{B}}{\mu_0}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_S \epsilon_0\vec{E}\cdot d\vec{s}$$

where S is any surface bounded by C.

Note that the same surface must be employed to evaluate the two surface integrals on the right side.

Also, the right hand screw rule must be used.

$\boxed{\frac{\vec{B}}{\mu_0} = \vec{H}}$ = Magnetic field intensity
Units are Amperes per meter

$$\vec{B} = \frac{\mu_0}{4\pi}\,\frac{I\,d\vec{l}\times\vec{R}}{R^3}$$

$$\frac{\vec{B}}{\mu_0} = \frac{1}{4\pi}\,\frac{I\,d\vec{l}\times\vec{R}}{R^3} \quad \frac{\text{Amp}\times m\times m}{m^3}$$

$\oint_C \vec{H}\cdot d\vec{l}$ is known as the magnetomotive force (mmf) in analogy with electromotive force (emf) for $\oint_C \vec{E}\cdot d\vec{l}$. However, it does not have a physical meaning as $\oint_C \vec{E}\cdot d\vec{l}$ does since magnetic force on a moving charge is directed perpendicular to the direction of motion of the charge and to the magnetic field.

$\int_S \vec{J}\cdot d\vec{s}$ = current crossing S due to actual motion of charges

$\boxed{\epsilon_0\vec{E} = \vec{D}}$ = displacement flux density
Units are coulombs per m^2

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$\epsilon_0 E = \frac{Q}{4\pi R^2} \quad \frac{\text{coulombs}}{m^2}$$

$\int_S \vec{D}\cdot d\vec{s}$ = Displacement flux crossing S, $C/m^2 \times m^2$, or C.

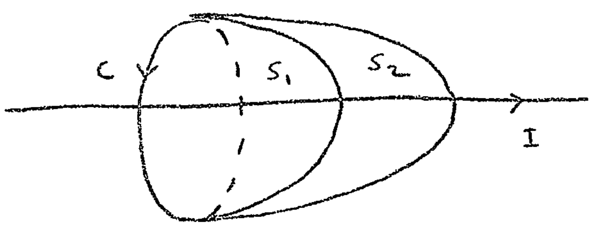
$\frac{d}{dt}\int_S \vec{D}\cdot d\vec{s}$ = time rate of change of displacement flux, $\frac{C}{\text{sec}}$, or Amps
= displacement current

$$\boxed{\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s}}$$

The magnetomotive force around a closed path C is equal to the total current, that is, the sum of the current due to actual flow of charges and the displacement current bounded by C, in accordance with the right hand screw rule.

Three cases to clarify Ampere's circuital law:

a) Infinitely long, current carrying wire:

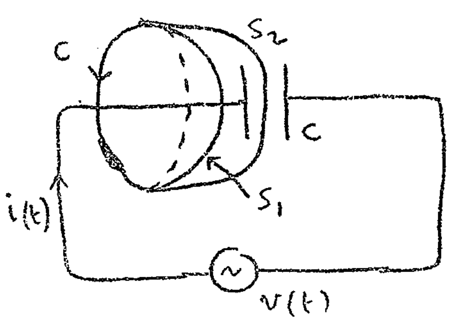


No displacement flux

$$\int_{S_1} \vec{J}\cdot d\vec{s} = \int_{S_2} \vec{J}\cdot d\vec{s} = I$$

$$\oint_C \vec{H}\cdot d\vec{l} = I$$

b) capacitor circuit (neglect fringing of field in the capacitor)

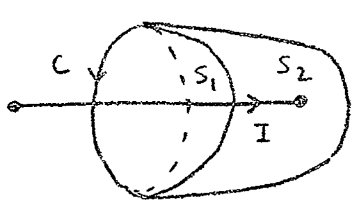


$$\int_{S_1} \vec{J}\cdot d\vec{s} = i \quad \text{but} \quad \int_{S_2} \vec{J}\cdot d\vec{s} = 0$$

$$\int_{S_1} \vec{D}\cdot d\vec{s} = 0 \quad \text{but} \quad \int_{S_2} \vec{D}\cdot d\vec{s} \neq 0$$

$\frac{d}{dt}\int_{S_2} \vec{D}\cdot d\vec{s}$ must be $= i$ so that $\oint_C \vec{H}\cdot d\vec{l}$ is unique.

c) Finitely long wire



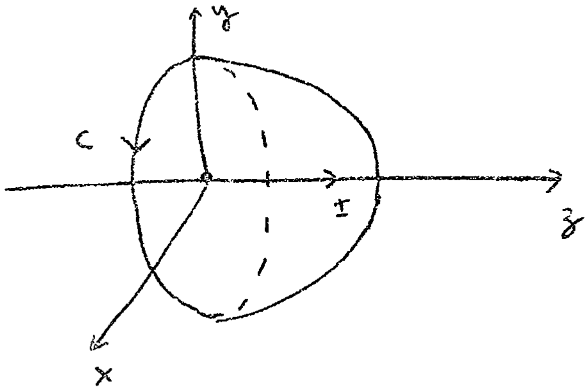
$$\int_{S_1} \vec{J}\cdot d\vec{s} = I \quad \text{and} \quad \frac{d}{dt}\int_{S_1} \vec{D}\cdot d\vec{s} \neq 0$$

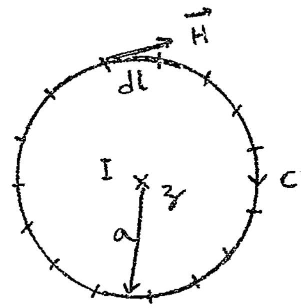
$$\int_{S_2} \vec{J}\cdot d\vec{s} = 0 \quad \text{and} \quad \frac{d}{dt}\int_{S_2} \vec{D}\cdot d\vec{s} \neq 0$$

$\int_{S_1} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{S_1} \vec{D}\cdot d\vec{s}$ must be $= \int_{S_2} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{S_2} \vec{J}\cdot d\vec{s}$

so that $\oint_S \vec{H}\cdot d\vec{l}$ is unique.

EXAMPLE : Let us consider an infinitely long, thin, straight wire situated along the z axis and carrying current in the z direction.





Then if we choose a circular contour c situated in the x y plane and centered at the origin, we have

$$\oint_C \vec{H} \cdot d\vec{l} = I$$

We know from elementary consideration of Biot Savart law and symmetry that $\vec{H}$ must be directed tangential to the circular path and must have same magnitude at all points on the circle. Hence, if we divide the circular path into several infinitesimal segments, then for each segment

$$\vec{H} \cdot d\vec{l} = H\, dl$$

$$\therefore \oint_C \vec{H} \cdot d\vec{l} = \oint_C H\, dl = H \oint_C dl = 2\pi a H$$

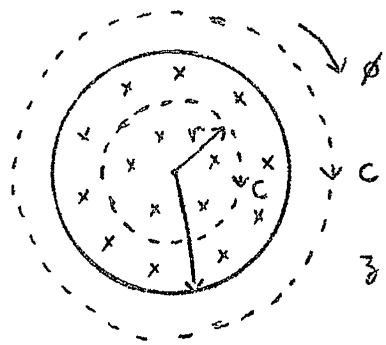
$$2\pi a H = I$$

$$H = \frac{I}{2\pi a}.$$

Thus the magnetic field due to the infinitely long wire is directed circular to the wire in the right hand sense and has a magnitude $\frac{I}{2\pi r}$ where r is the distance of the point from the wire.

EXAMPLE : Let us extend the previous example to the case of an infinitely long, straight, cylindrical wire of radius a carrying uniformly distributed current. Let the axis of the wire be the z axis so that

$$\vec{J} = \begin{cases} J_0 \, \vec{i}_z & \text{for } r<a \\ 0 & \text{for } r>a \end{cases}$$



From symmetry considerations,

$$\vec{H} = H_\phi(r)\,\vec{i}_\phi ,$$

that is, H must be directed everywhere circular to the axis of the wire and its magnitude must be uniform on cylindrical surfaces coaxial to the wire.

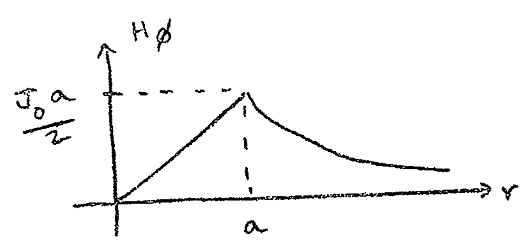
Then if we choose a circular path of radius r in a plane normal to the wire and centered on the axis of the wire, we have

$$\oint_C \vec{H}\cdot d\vec{l} = 2\pi r H_\phi$$

$$\int_S \vec{J}\cdot d\vec{s} = \begin{cases} J_0 \pi r^2 & \text{if } C \text{ is inside the wire, i.e., } r<a \\ J_0 \pi a^2 & \text{if } C \text{ is outside the wire, i.e., } r>a \end{cases}$$

$$\therefore\ 2\pi r H_\phi = \begin{cases} J_0 \pi r^2 & \text{for } r<a \\ J_0 \pi a^2 & \text{for } r>a \end{cases}$$

$$H_\phi = \begin{cases} \dfrac{J_0 r}{2} & \text{for } r<a \\ \dfrac{J_0 a^2}{2r} & \text{for } r>a \end{cases}$$

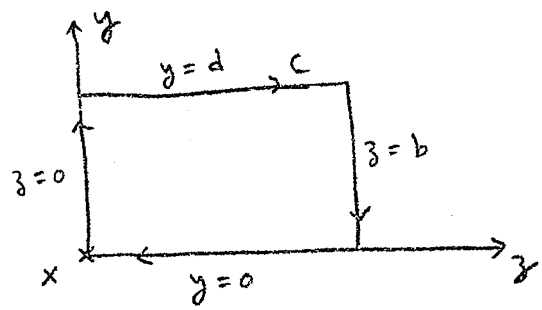


EXAMPLE: Let us consider a time varying electric field

$$\vec{E} = E_0 \, z \sin \omega t \; \vec{i}_x$$

and a rectangular loop in the yz plane as shown in the figure.

Then



$$\oint_C \vec{H} \cdot d\vec{l} = \frac{d}{dt} \int_S \vec{D} \cdot d\vec{s}$$

$$\int_S \vec{D} \cdot d\vec{s} = \int_{z=0}^{b} \int_{y=0}^{d} \epsilon_0 E_0 \, z \sin \omega t \; \vec{i}_x \cdot dy \, dz \; \vec{i}_x$$

$$= \epsilon_0 E_0 \sin \omega t \int_{z=0}^{b} \int_{y=0}^{d} z \, dy \, dz$$

$$= \epsilon_0 \frac{b^2 d}{2} E_0 \sin \omega t .$$

$$\oint_C \vec{H} \cdot d\vec{l} = \frac{d}{dt} \int_S \vec{D} \cdot d\vec{s}$$

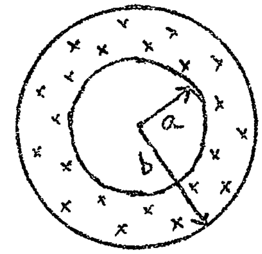
$$= \frac{d}{dt} \left[\epsilon_0 \frac{b^2 d}{2} E_0 \sin \omega t \right]$$

$$= \epsilon_0 \frac{b^2 d}{2} E_0 \, \omega \cos \omega t .$$

Homework Problem (due 9/20/74)

A hollow, infinitely long, cylindrical wire of inner radius a and outer radius b carries a uniformly distributed current in the z direction, i.e.,

$$\vec{J} = \begin{cases} 0 & \text{for } r < a \\ J_0 \vec{i}_z & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$



z axis into the paper

Find $\vec{H}$ everywhere.

Additional Problems

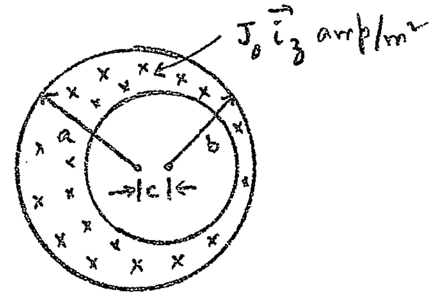
1. An infinitely long, cylindrical wire radius a, having its axis along the z axis carries a nonuniform current distribution given by

$$\vec{J} = \begin{cases} J_0 \frac{r}{a} \vec{i}_z & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

where J_0 is a constant. Find $\vec{H}$ everywhere

2. Current flows axially with uniform density J_0 amp/m² in the region between two infinitely long, parallel cylindrical surfaces of radii a and b (< a) and with their axes separated by a distance c (< a−b) as shown in the figure.

Find $\vec{H}$ in the current-free region inside the cylindrical surface of radius b.



GAUSS' LAW FOR THE ELECTRIC FIELD:

$$\boxed{\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv}$$

$\oint_S \vec{D} \cdot d\vec{s}$ = Displacement flux emanating from the closed surface S, i.e., the flux of $\vec{D}$ or $\epsilon_0 \vec{E}$.

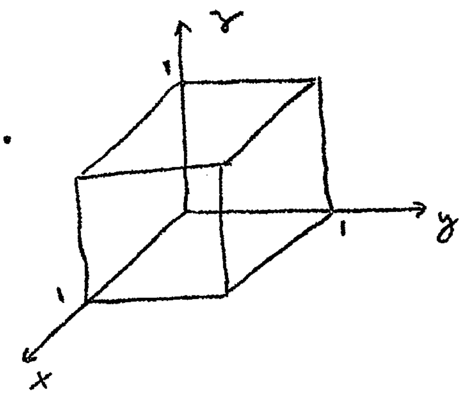
ρ = Volume charge density = $\lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$ C/m^3

$\int_V \rho \, dv$ = Volume charge contained in V.

EXAMPLE OF VOLUME INTEGRATION:

Let $\rho = x + y + z$ C/m^3

We wish to find the charge within the cube.



$$\int_V \rho \, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (x+y+z)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[xz + yz + \frac{z^2}{2} \right]_{z=0}^{1} dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[x + y + \frac{1}{2} \right] dx\, dy$$

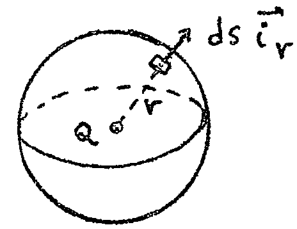
$$= \int_{x=0}^{1} \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_{y=0}^{1} dx = \int_{x=0}^{1} (x+1)\, dx$$

$$= \left[\frac{x^2}{2} + x \right]_0^1 = \frac{3}{2} \text{ Coulombs.}$$

This, in turn, mean that the displacement flux, $\oint_S \vec{D} \cdot d\vec{s}$, emanating from the cubical surface is $\frac{3}{2}$ C.

APPLICATIONS OF GAUSS LAW TO COMPUTE STATIC FIELDS:

EXAMPLE: Field due to point charge



From symmetry considerations,

$$\vec{E} = E_r(r)\,\vec{i}_r$$

$$\vec{D} = \epsilon_0 \vec{E} = D_r(r)\,\vec{i}_r$$

Then if we choose a spherical surface of radius r (Gaussian surface) and apply Gauss' law,

$$\oint_S \vec{D}\cdot d\vec{s} = \oint_S D_r\,\vec{i}_r \cdot ds\,\vec{i}_r = D_r \oint_S ds$$

$$= D_r\,(\text{area of the spherical surface})$$

$$= 4\pi r^2 D_r$$

$$\int_V \rho\, dv = \text{charge enclosed by } S = \text{point charge } Q.$$

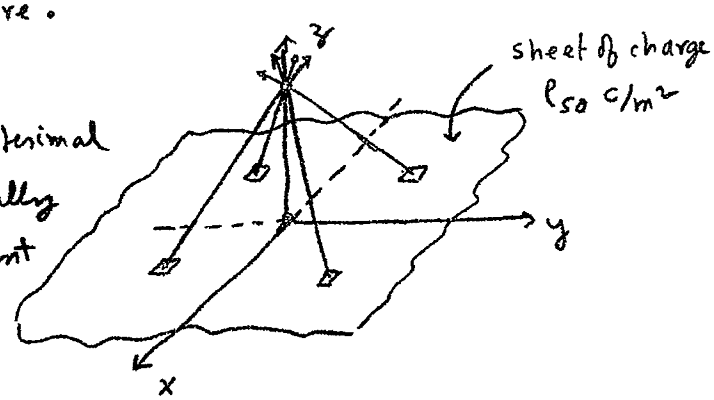
Thus, $4\pi r^2 D_r = Q$

$$D_r = \frac{Q}{4\pi r^2}, \quad E_r = \frac{D_r}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\vec{i}_r$$

EXAMPLE: Let us consider infinite sheet of charge in the xy plane and having uniform surface charge density, ρ_{s0} C/m². We wish to find the electric field everywhere.

First, we note that if we take a point on the z-axis and consider four infinitesimal charges on the sheet and symmetrically situated about the origin, the resultant field due to them is directed in the z direction.

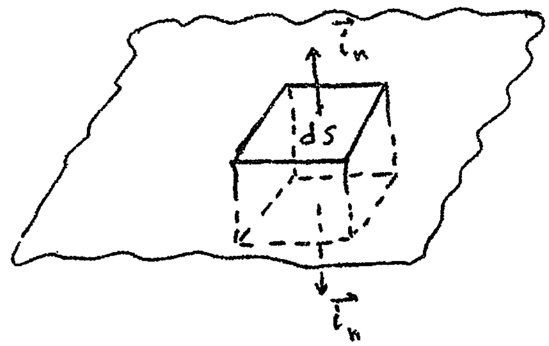


Since we can divide the sheet into several such symmetrical configurations of infinitesimal charges, the field due to the entire sheet charge is directed in the z direction. Furthermore, since this argument can be applied to any point, the field is directed everywhere above the sheet in the z direction and it can also be seen that it is independent of x and y. Likewise, below the sheet, it is directed in the negative z direction. Thus, we can say that

$$\vec{D} = \begin{cases} D_z(z)\,\vec{i}_z & \text{for } z>0 \\ -D_z(z)\,\vec{i}_z & \text{for } z<0 \end{cases} = D_n(z)\,\vec{i}_n$$

where $\vec{i}_n$ is unit vector directed normal to the sheet and away from it.

If we now consider a rectangular box symmetrically situated about the sheet of charge and apply Gauss' law,

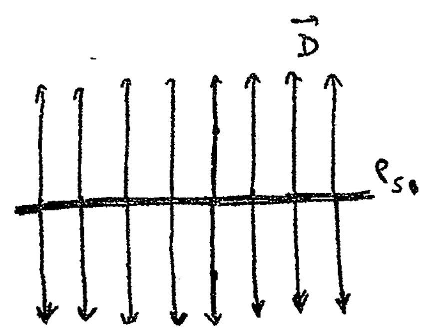


$$\oint_S \vec{D}\cdot\vec{ds} = \int_{\text{top surface}} \vec{D}\cdot\vec{ds} + \int_{\text{bottom surface}} \vec{D}\cdot\vec{ds} + \int_{\text{side surfaces}} \vec{D}\cdot\vec{ds}$$

$$= D_z\,ds + D_z\,ds + 0 = 2D_z\,ds$$

$\int_V \rho\,dv$ = surface charge enclosed by the box = $\rho_{s0}\,ds$

Thus, $2D_z\,ds = \rho_{s0}\,ds$ or $D_z = \dfrac{\rho_{s0}}{2}$

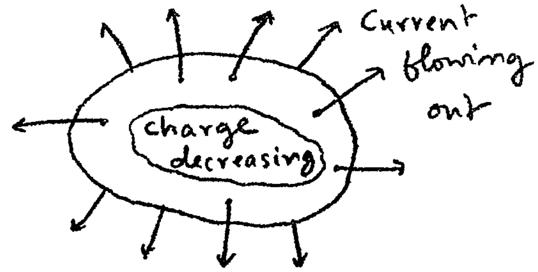


$$\vec{D} = \begin{cases} \dfrac{\rho_{s0}}{2}\,\vec{i}_z & \text{for } z>0 \\ -\dfrac{\rho_{s0}}{2}\,\vec{i}_z & \text{for } z<0 \end{cases} = \frac{\rho_{s0}}{2}\,\vec{i}_n$$

Dependency of Gauss' law for the electric field on Ampere's Circuital law and the law of conservation of charge:

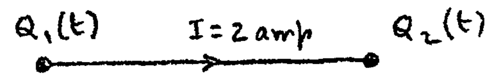
What is the law of conservation of charge?

It states that the current due to flow of charges emanating from a closed surface S is equal to the time rate of decrease of the charge within the volume V bounded by S.

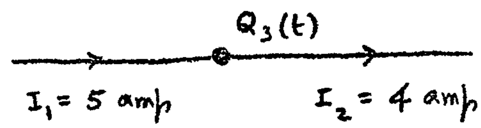


$$\oint_S \vec{J} \cdot d\vec{s} = -\frac{d}{dt}\int_V \rho \, dv$$

or $$\oint_S \vec{J} \cdot d\vec{s} + \frac{d}{dt}\int_V \rho \, dv = 0.$$



$$\frac{dQ_1}{dt} = -2, \quad \frac{dQ_2}{dt} = 2$$



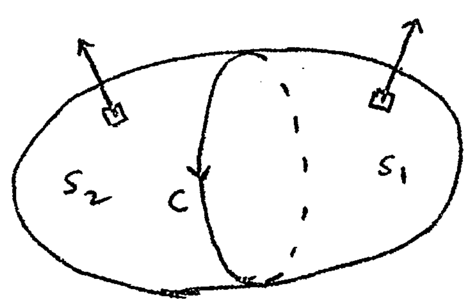
$$\frac{dQ_3}{dt} = 5 - 4 = 1$$

Applying Ampere's circuital law to two surfaces S_1 and S_2 bounded by C, we have

$$\oint_C \vec{H} \cdot d\vec{l} = \int_{S_1} \vec{J} \cdot d\vec{s}_1 + \frac{d}{dt}\int_{S_1} \vec{D} \cdot d\vec{s}_1$$

$$\oint_C \vec{H} \cdot d\vec{l} = -\int_{S_2} \vec{J} \cdot d\vec{s}_2 - \frac{d}{dt}\int_{S_2} \vec{D} \cdot d\vec{s}_2$$

$$\int_{S_1} \vec{J} \cdot d\vec{s}_1 + \frac{d}{dt}\int_{S_1} \vec{D} \cdot d\vec{s}_1 = -\int_{S_2} \vec{J} \cdot d\vec{s}_2 - \frac{d}{dt}\int_{S_2} \vec{D} \cdot d\vec{s}_2$$



$$\oint_{S_1+S_2} \vec{J} \cdot d\vec{s} + \frac{d}{dt}\oint_{S_1+S_2} \vec{D} \cdot d\vec{s} = 0$$

$$-\frac{d}{dt}\int_V \rho \, dv + \frac{d}{dt}\oint_S \vec{D} \cdot d\vec{s} = 0$$

$$\frac{d}{dt}\left[\oint_S \vec{D} \cdot d\vec{s} - \int_V \rho \, dv\right] = 0$$

$$\oint_S \vec{D} \cdot d\vec{s} - \int_V \rho \, dv = \text{constant with time} = 0$$

$$\therefore \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv \quad \text{which is Gauss' law.}$$

Homework Problem (due 9/23/74)

A spherical volume charge of uniform charge density ρ_0 c/m³ and having radius a meters is centered at the origin. Using Gauss' law for the electric field, find the electric field intensity everywhere.

Additional Problems:

1. Given $\rho = xyz$ C/m³, find the displacement flux emanating from the surface of the cube defined by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

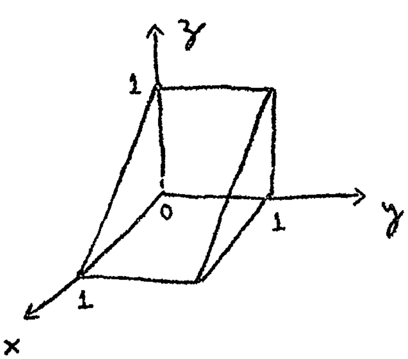
2. An infinitely long, uniform, line charge of density ρ_{L0} C/m is situated along the z axis. Using Gauss' law, find the electric field due to the line charge.

3. Given $\vec{D} = x\,\vec{i}_x$, find the volume charge density contained in the volume of the wedge-shaped box defined by the planes

$x = 0$, $x + z = 1$

$y = 0$, $y = 1$

$z = 0$



4. Repeat 3 for $\vec{D} = x^2\,\vec{i}_x$

5. Given $\vec{J} = y\,\vec{i}_y$, find the time rate of increase of charge contained in the volume specified in 3.

NEXT 5 LECTURES: MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

HOUR EXAM NO. 1: WEDNESDAY 9/25/74

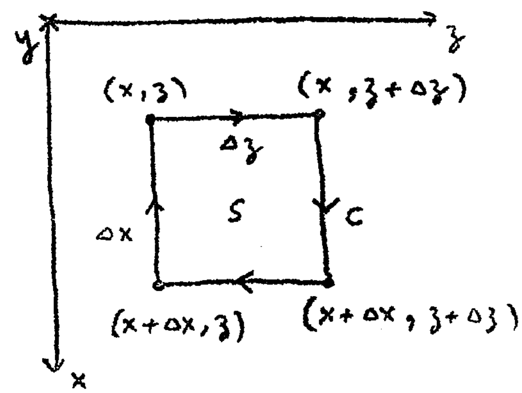
FARADAY'S LAW:

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{s}$$

Let us consider the simple case of

$$\vec{E} = E_x(z,t)\,\vec{i}_x$$

and apply Faraday's law in integral form to the rectangular path defined by the points (x,z), $(x, z+\Delta z)$, $(x+\Delta x, z+\Delta z)$ and $(x+\Delta x, z)$ in a constant y plane.



$$\oint_C \vec{E}\cdot d\vec{l} = \int_{(x,z)}^{(x,z+\Delta z)} \vec{E}\cdot d\vec{l} + \int_{(x,z+\Delta z)}^{(x+\Delta x, z+\Delta z)} \vec{E}\cdot d\vec{l} + \int_{(x+\Delta x, z+\Delta z)}^{(x+\Delta x, z)} \vec{E}\cdot d\vec{l} + \int_{(x+\Delta x, z)}^{(x,z)} \vec{E}\cdot d\vec{l}$$

$$= 0 + [E_x]_{z+\Delta z}\,\Delta x + 0 - [E_x]_z\,\Delta x$$

$$= \left\{[E_x]_{z+\Delta z} - [E_x]_z\right\}\Delta x$$

$$\int_S \vec{B}\cdot d\vec{s} = [B_y]_{(x,z)}\,\Delta x\,\Delta z$$

$$\therefore \left\{[E_x]_{z+\Delta z} - [E_x]_z\right\}\Delta x = -\frac{d}{dt}\left\{[B_y]_{(x,z)}\,\Delta x\,\Delta z\right\}$$

$$\frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\frac{\partial [B_y]_{(x,z)}}{\partial t}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = - \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{\partial [B_y]_{(x,z)}}{\partial t}$$

$$\boxed{\frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t}}$$

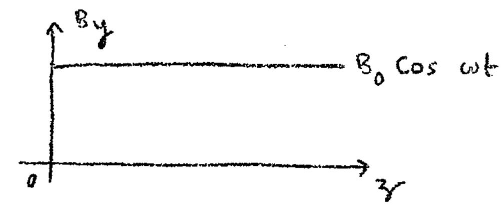
Faraday's law in differential form for the simple case of $\vec{E} = E_x(z,t)\,\vec{i}_x$.

A time varying B_y at a point results in an E_x at that point having a differential in the z direction.

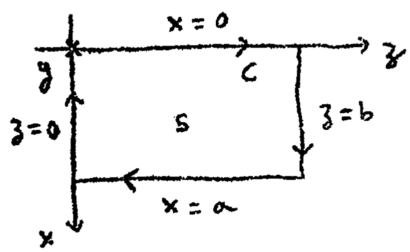
EXAMPLE: Given $\vec{B} = B_0 \cos \omega t \; \vec{i}_y$ and $\vec{E}$ has only an x component.

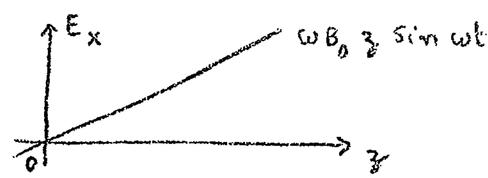
$$\frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t} = - \frac{\partial}{\partial t}(B_0 \cos \omega t) = \omega B_0 \sin \omega t$$

$$E_x = \omega B_0 z \sin \omega t \quad \propto z$$



Verification:





$$\oint_C \vec{E} \cdot \vec{dl} = \int_{z=0}^{b} [E_z]_{x=0}\, dz + \int_{x=0}^{a} [E_x]_{z=b}\, dx$$

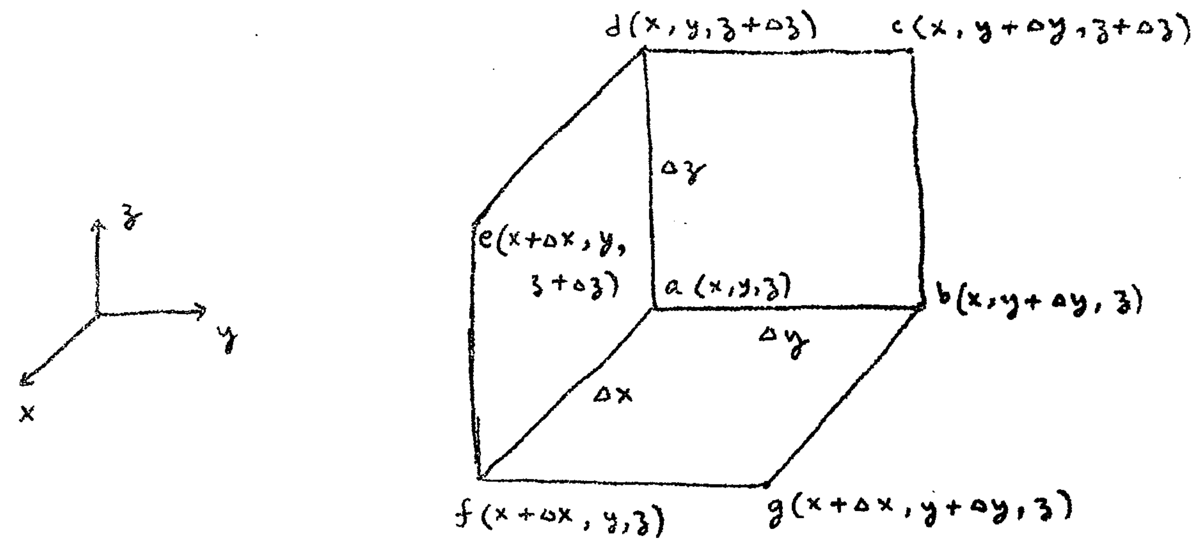
$$+ \int_{z=b}^{0} [E_z]_{x=a}\, dz + \int_{x=a}^{0} [E_x]_{z=0}\, dx$$

$$= 0 + [\omega B_0 b \sin \omega t]\, a + 0 + 0 = a b \omega B_0 \sin \omega t$$

$$= - \frac{d}{dt} [a b B_0 \cos \omega t] = - \frac{d}{dt} \int_S \vec{B} \cdot \vec{dS}$$

Faraday's Law in Differential Form for the General Case:

$$\vec{E} = E_x(x,y,z,t)\,\vec{i}_x + E_y(x,y,z,t)\,\vec{i}_y + E_z(x,y,z,t)\,\vec{i}_z$$



$$\oint_{abcda} \vec{E}\cdot d\vec{l} = [E_y]_{(x,z)}\,\Delta y + [E_z]_{(x,y+\Delta y)}\,\Delta z - [E_y]_{(x,z+\Delta z)}\,\Delta y - [E_z]_{(x,y)}\,\Delta z$$

$$\int_{abcd} \vec{B}\cdot d\vec{s} = [B_x]_{(x,y,z)}\,\Delta y\,\Delta z$$

$$\left\{[E_z]_{(x,y+\Delta y)} - [E_z]_{(x,y)}\right\}\Delta z - \left\{[E_y]_{(x,z+\Delta z)} - [E_y]_{(x,z)}\right\}\Delta y = -\frac{d}{dt}\left\{[B_x]_{(x,y,z)}\,\Delta y\,\Delta z\right\}$$

Dividing both sides by $\Delta y\,\Delta z$,

$$\frac{[E_z]_{(x,y+\Delta y)} - [E_z]_{(x,y)}}{\Delta y} - \frac{[E_y]_{(x,z+\Delta z)} - [E_y]_{(x,z)}}{\Delta z} = -\frac{\partial [B_x]_{(x,y,z)}}{\partial t}$$

Taking the limit on both sides as Δy and Δz tend to zero, we have

$$\lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{[E_z]_{(x,y+\Delta y)} - [E_z]_{(x,y)}}{\Delta y} - \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{[E_y]_{(x,z+\Delta z)} - [E_y]_{(x,z)}}{\Delta z}$$

$$= - \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{\partial [B_x]_{(x,y,z)}}{\partial t}$$

$$\boxed{\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = - \frac{\partial B_x}{\partial t}}$$

Similarly, applying Faraday's law to the closed path adefa, and proceeding in a similar manner, we get

$$[E_z]_{(x,y)} \Delta z + [E_x]_{(y,z+\Delta z)} \Delta x - [E_z]_{(x+\Delta x,y)} \Delta z - [E_x]_{(y,z)} \Delta x$$

$$= - \frac{d}{dt} \left\{ [B_y]_{(x,y,z)} \Delta z \, \Delta x \right\}$$

$$\lim_{\substack{\Delta z \to 0 \\ \Delta x \to 0}} \frac{[E_x]_{(y,z+\Delta z)} - [E_x]_{(y,z)}}{\Delta z} - \lim_{\substack{\Delta z \to 0 \\ \Delta x \to 0}} \frac{[E_z]_{(x+\Delta x,y)} - [E_z]_{(x,y)}}{\Delta x}$$

$$= - \lim_{\substack{\Delta z \to 0 \\ \Delta x \to 0}} \frac{\partial [B_y]_{(x,y,z)}}{\partial t}$$

$$\boxed{\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = - \frac{\partial B_y}{\partial t}}$$

Finally, applying Faraday's law to the closed path afgba, and proceeding in a similar manner, we obtain

$$\boxed{\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = - \frac{\partial B_z}{\partial t}}$$

Combining the three equations, we write

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{i}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\vec{i}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{i}_z$$

$$= -\frac{\partial B_x}{\partial t}\vec{i}_x - \frac{\partial B_y}{\partial t}\vec{i}_y - \frac{\partial B_z}{\partial t}\vec{i}_z$$

$$\begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial}{\partial t}\left(B_x\vec{i}_x + B_y\vec{i}_y + B_z\vec{i}_z\right)$$

$$\left(\vec{i}_x\frac{\partial}{\partial x} + \vec{i}_y\frac{\partial}{\partial y} + \vec{i}_y\frac{\partial}{\partial z}\right) \times \left(E_x\vec{i}_x + E_y\vec{i}_y + E_z\vec{i}_z\right) = -\frac{\partial \vec{B}}{\partial t}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ Faraday's law in differential form

$\vec{\nabla} \times \vec{E}$ is known as "del cross $\vec{E}$" or the "curl of $\vec{E}$".

<u>EXAMPLE</u>: $\vec{A} = y\vec{i}_x - x\vec{i}_y$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \vec{i}_x(0) + \vec{i}_y(0) + \vec{i}_z(-1-1) = -2\vec{i}_z.$$

<u>Homework Problem</u> (due 9/27/74)

Given $\vec{E} = 10\cos(6\pi \times 10^8 t - 2\pi z)$ V/m, find $\vec{B}$.

AMPERE'S CIRCUITAL LAW:

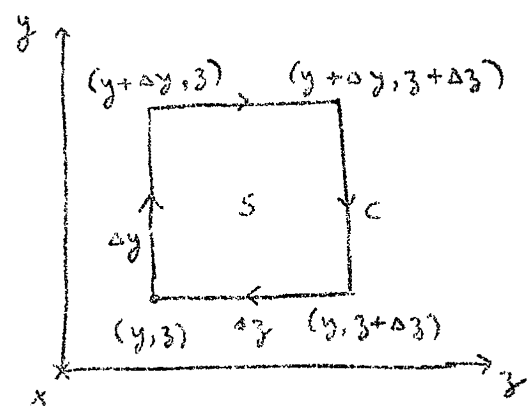
Ampere's circuital law in integral form is given by

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{d}{dt} \int_S \vec{D} \cdot d\vec{S}$$

Let us consider the simple case in which

$$\vec{H} = H_y(z, t)\, \vec{i}_y$$

and apply Ampere's circuital law to the infinitesimal rectangular path shown in the figure.



$$\oint_C \vec{H} \cdot d\vec{l} = \int_{(y,z)}^{(y+\Delta y, z)} \vec{H} \cdot d\vec{l} + \int_{(y+\Delta y, z)}^{(y+\Delta y, z+\Delta z)} \vec{H} \cdot d\vec{l}$$

$$+ \int_{(y+\Delta y, z+\Delta z)}^{(y, z+\Delta z)} + \int_{(y, z+\Delta z)}^{(y,z)} \vec{H} \cdot d\vec{l}$$

$$= [H_y]_z \,\Delta y + 0 - [H_y]_{z+\Delta z}\,\Delta y + 0$$

$$= -\left\{ [H_y]_{z+\Delta z} - [H_y]_z \right\} \Delta y$$

$$\int_S \vec{J}\cdot d\vec{s} = [J_x]_{(y,z)}\, \Delta y\, \Delta z$$

$$\frac{d}{dt}\int_S \vec{D}\cdot d\vec{s} = \frac{d}{dt}\left\{ [D_x]_{(y,z)}\, \Delta y\, \Delta z \right\} = \frac{\partial [D_x]_{(y,z)}}{\partial t}\, \Delta y\, \Delta z$$

Thus we have

$$-\left\{[H_y]_{z+\Delta z} - [H_y]_z\right\}\Delta y = \left[J_x + \frac{\partial D_x}{\partial t}\right]_{(y,z)} \Delta y\, \Delta z$$

Dividing throughout by $\Delta y\, \Delta z$ and taking the limit as Δy and Δz tend to zero, we obtain

$$\lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \frac{[H_y]_{z+\Delta z} - [H_y]_z}{\Delta z} = - \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \left[J_x + \frac{\partial D_x}{\partial t}\right]_{(y,z)}$$

or,

$$\boxed{\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t}}$$

Ampere's circuital law in differential form for the simple case of $\vec{H} = H_y(z,t)\,\vec{i}_y$.

Recall that Faraday's law in differential form for the simple case of $\vec{E} = E_x(z,t)\,\vec{i}_x$ was found to be

$$\boxed{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}}$$

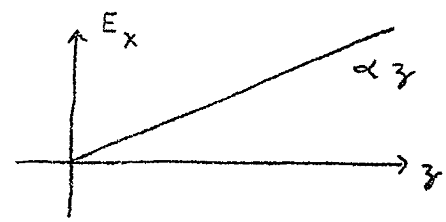
We shall find shortly that these two equations together give rise to uniform plane wave propagation.

EXAMPLE: Given $\vec{E} = E_0 z \sin \omega t \, \vec{i}_x$, $\vec{J} = 0$ and $\vec{B}$ has a y component only. Then

$$\frac{\partial H_y}{\partial z} = -J_x - \frac{\partial D_x}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 E_0 z \sin \omega t) = -\omega \epsilon_0 E_0 z \cos \omega t$$

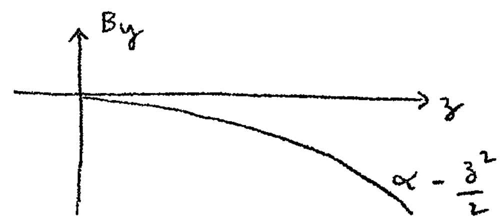
$$H_y = -\omega \epsilon_0 E_0 \frac{z^2}{2} \cos \omega t$$

$$B_y = \mu_0 H_y = -\omega \mu_0 \epsilon_0 E_0 \frac{z^2}{2} \cos \omega t.$$

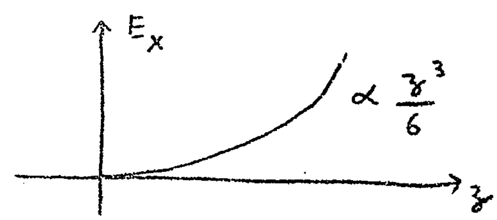


Proceeding further, we have

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = \omega^2 \mu_0 \epsilon_0 E_0 \frac{z^2}{2} \sin \omega t$$



$$E_x = \omega^2 \mu_0 \epsilon_0 E_0 \frac{z^3}{6} \sin \omega t$$



Note that we do not get the same E_x we started with in the example.

This means that the combinations of E_x and B_y we have here do not satisfy simultaneously the two Maxwell's equations in differential form.

HOMEWORK PROBLEM (due 10/2/74)

Given $\vec{B} = \frac{10^{-7}}{3} \cos(6\pi \times 10^8 t - 2\pi z) \, \vec{i}_y$ wb/m², find $\vec{E}$.

GENERAL CASE:

For the general case of

$$\vec{E} = E_x(x,y,z,t)\,\vec{i}_x + E_y(x,y,z,t)\,\vec{i}_y + E_z(x,y,z,t)\,\vec{i}_z$$

we found that

$$\oint_C \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds} \qquad \text{gives}$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

We can proceed in a similar manner and derive the differential form of Ampere's circuital law for the general case of

$$\vec{H} = H_x(x,y,z,t)\,\vec{i}_x + H_y(x,y,z,t)\,\vec{i}_y + H_z(x,y,z,t)\,\vec{i}_z\,.$$

But, it is very easy to see from analogy that

$$\oint_C \vec{H}\cdot\vec{dl} = \int_S \vec{J}\cdot\vec{ds} + \frac{d}{dt}\int_S \vec{D}\cdot\vec{ds} \qquad \text{gives}$$

$$\boxed{\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}}$$

where

$$\vec{\nabla}\times\vec{H} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

BASIC DEFINITION OF CURL:

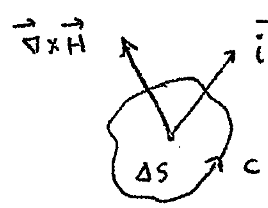
Let us for simplicity consider Ampere's circuital law in differential form without the displacement current density term, i.e.,

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Then, applying it to an infinitesimal surface $\vec{\Delta S}$, we have

$$(\vec{\nabla} \times \vec{H}) \cdot \vec{\Delta S} = \vec{J} \cdot \vec{\Delta S}$$

But from Ampere's circuital law in integral form without the displacement current term



$$\oint_c \vec{H} \cdot \vec{dl} = \vec{J} \cdot \vec{\Delta S}$$

$$\therefore \; (\vec{\nabla} \times \vec{H}) \cdot \vec{\Delta S} = \oint_c \vec{H} \cdot \vec{dl}$$

$$(\vec{\nabla} \times \vec{H}) \cdot \Delta S \, \vec{i}_n = \oint_c \vec{H} \cdot \vec{dl}$$

$$(\vec{\nabla} \times \vec{H}) \cdot \vec{i}_n = \frac{\oint_c \vec{H} \cdot \vec{dl}}{\Delta S} = |\vec{\nabla} \times \vec{H}| \cdot \cos \alpha \angle^{\vec{\nabla}\times\vec{H}}_{\vec{i}_n}$$

$$|\vec{\nabla} \times \vec{H}| = \left[\frac{\oint_c \vec{H} \cdot \vec{dl}}{\Delta S} \right]_{max}$$

$$\vec{\nabla} \times \vec{H} = \left[\frac{\oint_c \vec{H} \cdot \vec{dl}}{\Delta S} \right]_{max} \vec{i}_n$$

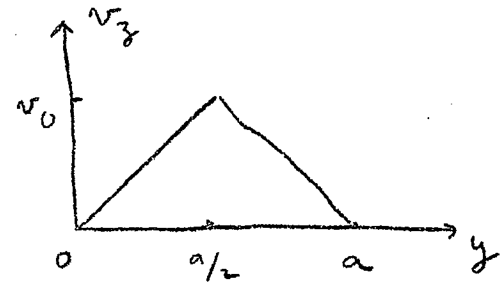
where $\vec{i}_n$ the unit normal vector to ΔS for which the maximum value occurs. Here it is parallel to $\vec{J}$. More exactly,

$$\boxed{\vec{\nabla} \times \vec{H} = \lim_{\Delta S \to 0} \left[\frac{\oint_c \vec{H} \cdot \vec{dl}}{\Delta S} \right]_{max} \vec{i}_n}$$

PHYSICAL INTERPRETATION OF CURL:

Let us consider a stream of rectangular crosssection carrying water in the z direction with a velocity independent of height but increasing uniformly from a value of zero at the banks to a maximum value v_0 at the center.

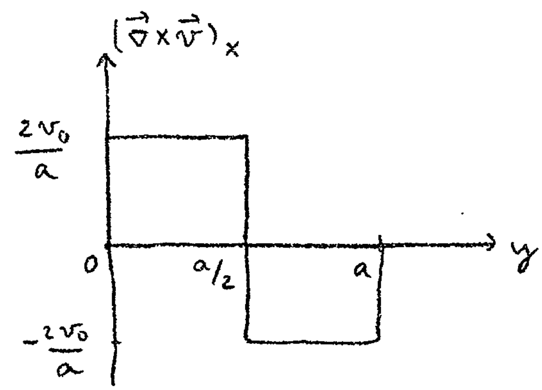
$$\vec{v} = \begin{cases} \dfrac{2v_0}{a} y \, \vec{i}_z & \text{for } 0 < y < \dfrac{a}{2} \\[2ex] 2v_0\left(1 - \dfrac{y}{a}\right)\vec{i}_z & \text{for } \dfrac{a}{2} < y < a \end{cases}$$

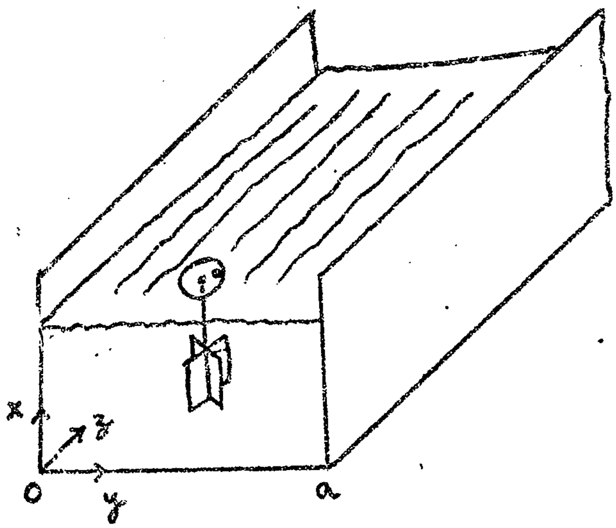


$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix}$$

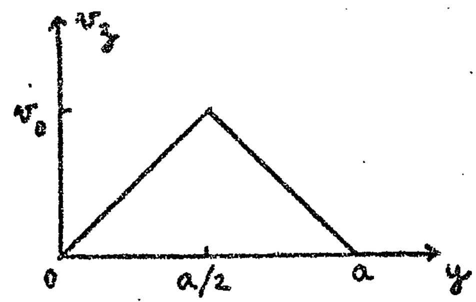
$$= \frac{\partial v_z}{\partial y} \vec{i}_x$$

$$= \begin{cases} \dfrac{2v_0}{a} \vec{i}_x & \text{for } 0 < y < \dfrac{a}{2} \\[2ex] -\dfrac{2v_0}{a} \vec{i}_x & \text{for } \dfrac{a}{2} < y < a \end{cases}$$

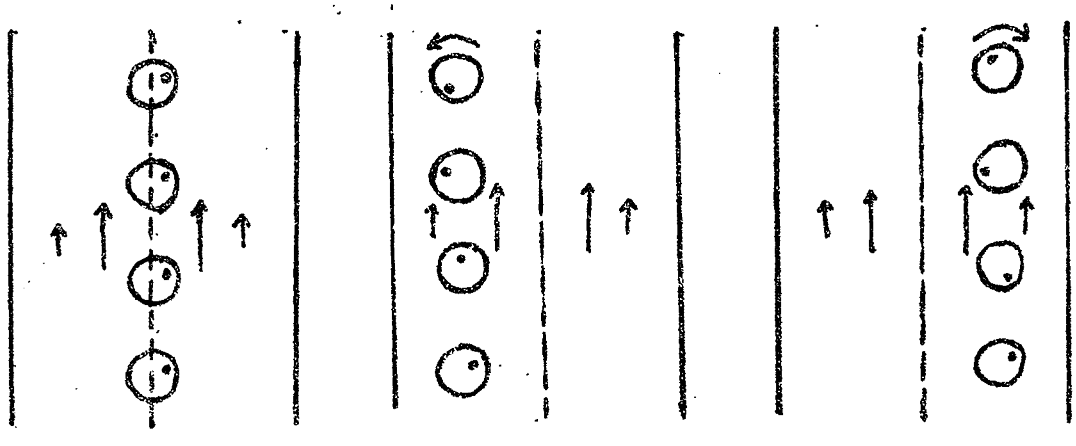




(a)



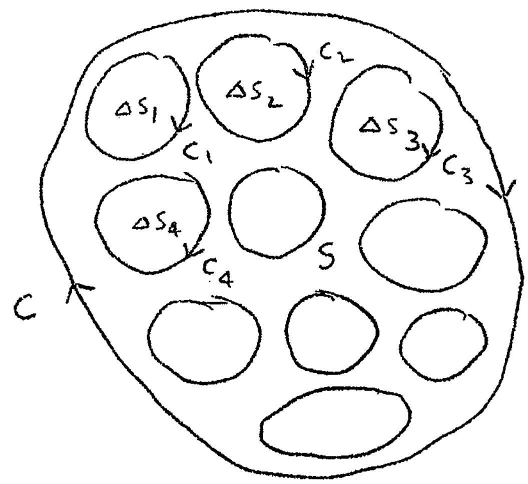
(b)



(c) (d) (e)

STOKES' THEOREM

Let us consider an arbitrary surface S and divide it into a large number of infinitesimal surfaces $\Delta S_1, \Delta S_2, \Delta S_3, \ldots$ bounded by the contours $c_1, c_2, c_3, \ldots$ Then



$$(\vec{\nabla} \times \vec{H})_1 \cdot \vec{\Delta S}_1 = \vec{J}_1 \cdot \vec{\Delta S}_1 = \oint_{c_1} \vec{H} \cdot \vec{dl}$$

$$(\vec{\nabla} \times \vec{H})_2 \cdot \vec{\Delta S}_2 = \vec{J}_2 \cdot \vec{\Delta S}_2 = \oint_{c_2} \vec{H} \cdot \vec{dl}$$

... and so on.

Adding up, we get

$$(\vec{\nabla} \times \vec{H})_1 \cdot \vec{\Delta S}_1 + (\vec{\nabla} \times \vec{H})_2 \cdot \vec{\Delta S}_2 + \cdots = \oint_{c_1} \vec{H} \cdot \vec{dl} + \oint_{c_2} \vec{H} \cdot \vec{dl} + \cdots$$

$$\sum_{j=1}^{n} (\vec{\nabla} \times \vec{H})_j \cdot \vec{\Delta S}_j = \sum_j \oint_{c_j} \vec{H} \cdot \vec{dl}$$

In the limit that $n \to \infty$, we have

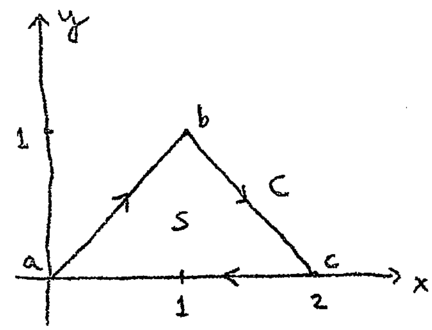
$$\boxed{\int_S (\vec{\nabla} \times \vec{H}) \cdot \vec{ds} = \oint_C \vec{H} \cdot \vec{dl}}$$ Stokes' theorem

Stoke's theorem enables us to replace the line integral of a vector by the surface integral of the curl of that vector and vice versa.

EXAMPLE: Let us verify Stoke's theorem by considering

$$\vec{A} = y\,\vec{i}_x - x\,\vec{i}_y$$

and the closed path c shown in the figure.



Evaluating the line integral first, we have

$$\vec{A}\cdot d\vec{l} = (y\,\vec{i}_x - x\,\vec{i}_y)\cdot(dx\,\vec{i}_x + dy\,\vec{i}_y)$$

$$= y\,dx - x\,dy$$

From a to b, $y = x$, $dy = dx$

$$\vec{A}\cdot d\vec{l} = x\,dx - x\,dx = 0$$

$$\int_a^b \vec{A}\cdot d\vec{l} = 0$$

From b to c, $y = 2 - x$, $dy = -dx$

$$\vec{A}\cdot d\vec{l} = (2-x)\,dx + x\,dx = 2\,dx$$

$$\int_b^c \vec{A}\cdot d\vec{l} = \int_{x=1}^{2} 2\,dx = 2$$

From c to a, $y = 0$, $dy = 0$, $\vec{A}\cdot d\vec{l} = 0$, $\int_c^a \vec{A}\cdot d\vec{l} = 0.$

$$\therefore \oint_c \vec{A}\cdot d\vec{l} = \int_a^b \vec{A}\cdot d\vec{l} + \int_b^c \vec{A}\cdot d\vec{l} + \int_c^a \vec{A}\cdot d\vec{l} = 0 + 2 + 0 = 2.$$

To evaluate the surface integral, we note that

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right]\vec{i}_z = -2\,\vec{i}_z.$$

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \int_S -2\vec{i}_z \cdot (-ds\,\vec{i}_z)$$

$$= \int_S 2\,ds = 2\int ds$$

$$= 2 \text{ (area of the triangle)}$$

$$= 2 \times 1 = 2$$

Thus Stokes' theorem is verified.

HOMEWORK PROBLEM (due 10/4/74)

For the vector $\vec{A} = yz\,\vec{i}_x + zx\,\vec{i}_y + xy\,\vec{i}_z$, use Stokes' theorem to show that $\oint_C \vec{A} \cdot d\vec{l}$ is zero for any closed path C.

Then evaluate $\int \vec{A} \cdot d\vec{l}$ from the origin to the point (1, 1, 1) along the curve $x = \sqrt{2}\sin t$, $y = \sqrt{2}\sin t$, $z = (4/\pi)t$.

GAUSS' LAW FOR THE ELECTRIC FIELD IN DIFFRENTIAL FORM:

Gauss' law for the electric field in integral form is given by

$$\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho \, dv$$

Let us consider a rectangular box of infinitesimal sides $\Delta x, \Delta y$ and Δz and defined by the six surfaces $x = x$, $x = x+\Delta x$, $y = y$, $y = y+\Delta y$, $z = z$ and $z = z+\Delta z$. Evaluating the left side of Gauss' law, that is, the displacement flux over the surface of this box, we obtain

$$\int \vec{D}\cdot d\vec{s} = -[D_x]_x \,\Delta y\,\Delta z \quad \text{for the surface } x = x$$

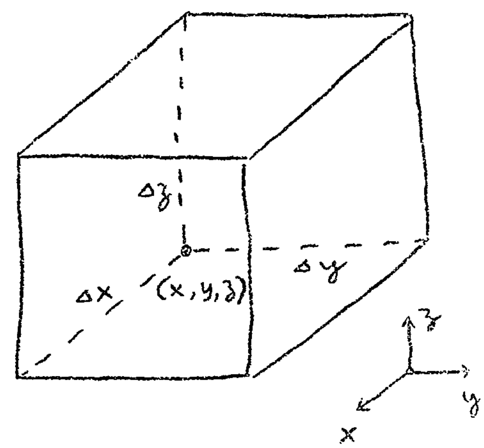
$$\int \vec{D}\cdot d\vec{s} = [D_x]_{x+\Delta x} \,\Delta y\,\Delta z \quad \text{for the surface } x = x+\Delta x$$

$$\int \vec{D}\cdot d\vec{s} = -[D_y]_y \,\Delta z\,\Delta x \quad \text{for the surface } y = y$$

$$\int \vec{D}\cdot d\vec{s} = [D_y]_{y+\Delta y} \,\Delta z\,\Delta x \quad \text{for the surface } y = y+\Delta y$$

$$\int \vec{D}\cdot d\vec{s} = -[D_z]_z \,\Delta x\,\Delta y \quad \text{for the surface } z = z$$

$$\int \vec{D}\cdot d\vec{s} = [D_z]_{z+\Delta z} \,\Delta x\,\Delta y \quad \text{for the surface } z = z+\Delta z$$



Adding up, we obtain the total displacement flux emanating from the rectangular box as

$$\oint \vec{D}\cdot d\vec{s} = \left\{[D_x]_{x+\Delta x} - [D_x]_x\right\}\Delta y\,\Delta z + \left\{[D_y]_{y+\Delta y} - [D_y]_y\right\}\Delta z\,\Delta x$$

$$+ \left\{[D_z]_{z+\Delta z} - [D_z]_z\right\}\Delta x\,\Delta y$$

The right side of Gauss' law in integral form, that is, the charge enclosed by the box is given by

$$\int_V \rho \, dv = \rho(x,y,z) \cdot \Delta x \, \Delta y \, \Delta z$$

Thus we have

$$\left\{ [D_x]_{x+\Delta x} - [D_x]_x \right\} \Delta y \, \Delta z + \left\{ [D_y]_{y+\Delta y} - [D_y]_y \right\} \Delta z \, \Delta x$$

$$+ \left\{ [D_z]_{z+\Delta z} - [D_z]_z \right\} \Delta x \, \Delta y = \rho(x,y,z) \cdot \Delta x \, \Delta y \, \Delta z$$

Dividing both sides by the volume $\Delta x \, \Delta y \, \Delta z$ and taking the limit as the volume shrinks to zero, that is, as $\Delta x, \Delta y, \Delta z \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \lim_{\Delta y \to 0} \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \rho(x,y,z)$$

or

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho}$$

This can be written as

$$\left(\vec{i}_x \frac{\partial}{\partial x} + \vec{i}_y \frac{\partial}{\partial y} + \vec{i}_z \frac{\partial}{\partial z} \right) \cdot \left(D_x \vec{i}_x + D_y \vec{i}_y + D_z \vec{i}_z \right) = \rho$$

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

$\vec{\nabla} \cdot \vec{D}$ is known as "del dot $\vec{D}$" or the divergence of $\vec{D}$.

EXAMPLE: Given $\vec{A} = xy\,\vec{i}_x + yz\,\vec{i}_y + zx\,\vec{i}_z$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

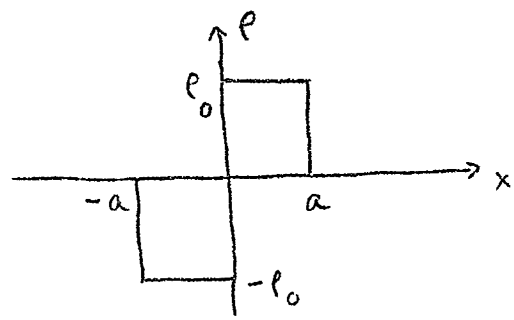
$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x = x + y + z.$$

EXAMPLE: Let us consider a volume charge distribution given by

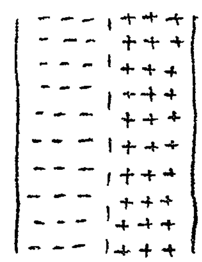
$$\rho = \begin{cases} -\rho_0 & \text{for } -a < x < 0 \\ \rho_0 & \text{for } 0 < x < a \end{cases}$$

where ρ_0 is a constant.



Since the charge density is independent of y and z, the field will also be independent of y and z. Hence

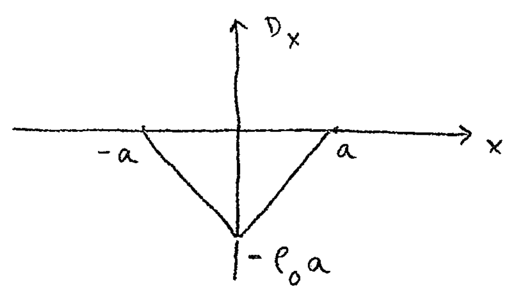
$$\frac{\partial D_y}{\partial y} = \frac{\partial D_z}{\partial z} = 0.$$



Thus, $$\frac{\partial D_x}{\partial x} = \begin{cases} -\rho_0 & \text{for } -a < x < 0 \\ \rho_0 & \text{for } 0 < x < a \end{cases}$$

$$D_x = \int \rho\, dx$$

Evaluating D_x graphically, we obtain the result shown in the sketch to the right.



GAUSS' LAW FOR THE MAGNETIC FIELD IN DIFFERENTIAL FORM:

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

Applying this integral form to the rectangular box and proceeding in the same manner as for the Gauss' law for the electric field, we get

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

$$\boxed{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0}$$

LAW OF CONSERVATION OF CHARGE IN DIFFERENTIAL FORM:

$$\oint_S \vec{J} \cdot d\vec{s} + \frac{d}{dt} \int_V \rho \, dv = 0$$

Again using the same technique as for the two Gauss' laws, we obtain

$$\boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is also known as the Continuity Equation.

$$\boxed{\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0}$$

BASIC DEFINITION OF DIVERGENCE

Applying Gauss' law in differential form

$$\vec{\nabla} \cdot \vec{D} = \rho$$

to an infinitesimal volume Δv, we have

$$(\vec{\nabla} \cdot \vec{D})\, dv = \rho\, \Delta v$$

But according to Gauss' law in integral form,

$$\oint_S \vec{D} \cdot d\vec{s} = \rho\, \Delta v$$

$$\therefore (\vec{\nabla} \cdot \vec{D})\, \Delta v = \oint_S \vec{D} \cdot d\vec{s}$$

$$\vec{\nabla} \cdot \vec{D} = \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta v}$$

This becomes exact in the limit $\Delta v \to 0$. Thus

$$\boxed{\vec{\nabla} \cdot \vec{D} = \lim_{\Delta v \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta v}}$$

DIVERGENCE THEOREM :

Let us consider a large volume V and divide it into a number of infinitesimal volumes $\Delta v_1, \Delta v_2, \Delta v_3, \ldots$ bounded by surfaces $S_1, S_2, S_3, \ldots$ Then

$$(\vec{\nabla} \cdot \vec{D})_1\, \Delta v_1 = \oint_{S_1} \vec{D} \cdot d\vec{s}$$

$$(\vec{\nabla} \cdot \vec{D})_2\, \Delta v_2 = \oint_{S_2} \vec{D} \cdot d\vec{s}$$

$$(\vec{\nabla} \cdot \vec{D})_3\, \Delta v_3 = \oint_{S_3} \vec{D} \cdot d\vec{s}$$

and so on.

Adding up, we get

$$(\vec{\nabla}\cdot\vec{D})_1 \Delta v_1 + (\vec{\nabla}\cdot\vec{D})_2 \Delta v_2 + \cdots = \oint_{S_1} \vec{D}\cdot\vec{ds} + \oint_{S_2} \vec{D}\cdot\vec{ds} + \cdots$$

$$\sum_{j=1}^{n} (\vec{\nabla}\cdot\vec{D})_j \Delta v_j = \sum_{j=1}^{n} \oint_{S_j} \vec{D}\cdot\vec{ds}$$

In the limit that $n \to \infty$, we have

$$\boxed{\int_V (\vec{\nabla}\cdot\vec{D})\, dv = \oint_S \vec{D}\cdot\vec{ds}}$$

This is known as the divergence theorem. It permits the replacement of the closed surface integral of a vector by the volume integral of the divergence of that vector.

<u>HOMEWORK PROBLEM</u> (due 10/7/74)

Given $\vec{A} = 3x\,\vec{i}_x + (y-3)\,\vec{i}_y + (2+z)\,\vec{i}_z$.

Verify the divergence theorem by considering the rectangular box bounded by the planes

$x = 0, \quad x = 1$

$y = 0, \quad y = 1$

$z = 0, \quad z = 1$

UNIFORM PLANE WAVE PROPAGATION

Let us consider the simple case of

$$\vec{E} = E_x(z,t)\,\vec{i}_x$$

$$\vec{H} = H_y(z,t)\,\vec{i}_y$$

Then, Faraday's and Ampere's circuital laws in differential form are given by

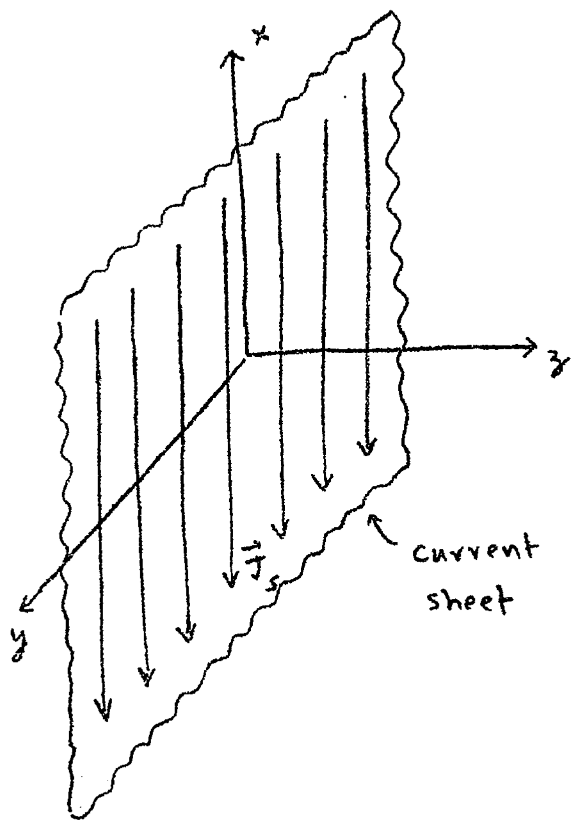
$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad \text{— (1)}$$

$$\frac{\partial H_y}{\partial z} = -\left[J_x + \frac{\partial D_x}{\partial t}\right] \qquad \text{— (2)}$$

We shall use these equations to find the electromagnetic field due to a sinusoidally time varying infinite plane current sheet characterized by

$$\vec{J}_S = -J_{S0}\cos\omega t\,\vec{i}_x \quad \text{amp/m}$$

and occupying the xy plane as shown in the figure.



we should however note that J_x in (2) represents a volume current density (amp/m²) whereas we now have a surface current density. Also, in the space on either side of the current sheet, there is no current and eqs. (1) and (2) reduce to

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t} \qquad -\ (3)$$

$$\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} = -\epsilon_0 \frac{\partial E_x}{\partial t} \qquad -\ (4)$$

we have to solve these equations to obtain the solution for the electromagnetic field on either side of the current sheet. To get a start on the solution, however, we consider the surface current distribution and find the magnetic field immediately adjacent to the current sheet and on either side of it by using Ampere's circuital law in integral form

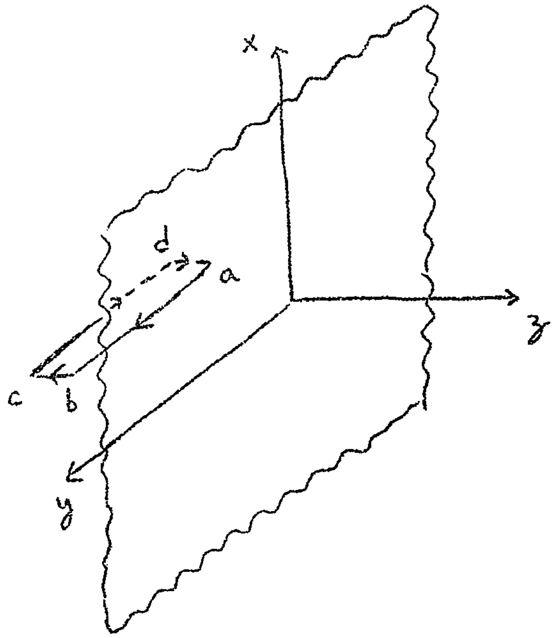
$$\oint_C \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot \vec{ds} + \frac{d}{dt} \int \vec{D} \cdot \vec{ds}$$

In order to do this we deliberately choose a rectangular path abcda lying parallel to the yz plane and with the sides ab and cd lying immediately adjacent to the current sheet and on either side of it.

$$\oint_c \vec{H}\cdot d\vec{l} = \int_a^b \vec{H}\cdot d\vec{l} + \int_b^c \vec{H}\cdot d\vec{l} + \int_c^d \vec{H}\cdot d\vec{l} + \int_d^a \vec{H}\cdot d\vec{l}$$

$$= [H_y]_{ab}\, ab + 0 - [H_y]_{cd}\, cd + 0$$

$$= 2\,[H_y]_{ab}\, ab$$



since from symmetry considerations,

$$[H_y]_{cd} = -[H_y]_{ab}.$$

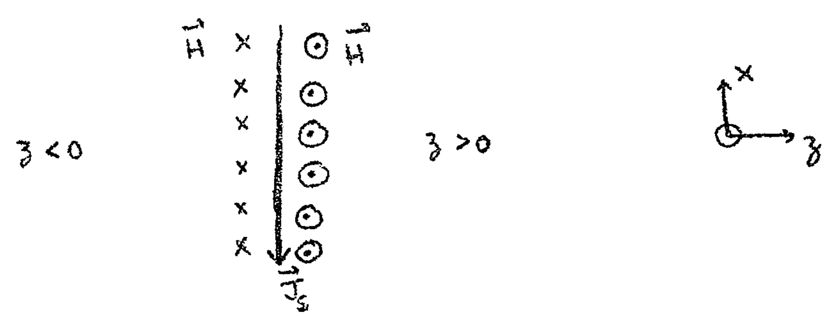
$$\int_s \vec{J}\cdot d\vec{s} = \int_{abcd} \vec{J}\cdot d\vec{s} = J_{so}\cos\omega t \cdot (ab)$$

$$\int_s \vec{D}\cdot d\vec{s} = \int_{abcd} \vec{D}\cdot d\vec{s} = 0$$

$$\therefore\quad 2\,[H_y]_{ab}\, ab = J_{so}\cos\omega t\cdot(ab)$$

or $$[H_y]_{ab} = \frac{J_{so}}{2}\cos\omega t \qquad \text{— (5)}$$

and $$[H_y]_{cd} = -\frac{J_{so}}{2}\cos\omega t \qquad \text{— (6)}$$



Considering now the regions on either side of the current sheet, we need to carry out the simultaneous solution of

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t} \qquad \text{— (3)}$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} \qquad \text{— (4)}$$

Differentiating (3) with respect to z and using (4), we obtain

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right) = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right) = -\mu_0 \frac{\partial}{\partial t}\left(-\epsilon_0 \frac{\partial E_x}{\partial t}\right)$$

or $\boxed{\dfrac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \dfrac{\partial^2 E_x}{\partial t^2}}$ known as the <u>wave equation</u> — (7)

To solve (7), which is a partial differential equation involving two variables z and t, we make use of the "separation of variables" technique. This consists of assuming

$$E_x(z,t) = Z(z) \cdot T(t)$$

$T(t)$ must however be of the form $A \cos \omega t + B \sin \omega t$ in view of the sinusoidally time varying nature of the source. Thus

$$E_x(z,t) = Z(z) \cdot [A \cos \omega t + B \sin \omega t] \qquad \text{— (8)}$$

Substituting (8) into (5), we obtain

$$Z''(z) \cdot [A \cos \omega t + B \sin \omega t] = \mu_0 \epsilon_0 Z(z) \cdot [-\omega^2 A \cos \omega t - \omega^2 B \sin \omega t]$$

or $$Z'' = -\omega^2 \mu_0 \epsilon_0 Z$$

$$Z(z) = A' \cos \omega\sqrt{\mu_0\epsilon_0}\, z + B' \sin \omega\sqrt{\mu_0\epsilon_0}\, z \qquad \text{— (9)}$$

Defining $\boxed{\beta = \omega\sqrt{\mu_0\epsilon_0}}$ and substituting (9) into (8), we get

$$E_x = (A' \cos \beta z + B' \sin \beta z)(A \cos \omega t + B \sin \omega t)$$

$$= C \cos \beta z \cos \omega t + D \cos \beta z \sin \omega t$$

$$+ C' \sin \beta z \cos \omega t + D' \sin \beta z \sin \omega t \qquad \text{— (10)}$$

The corresponding solution for H_y can be obtained by using (3) or (4). Thus using (4),

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \left[-\omega C \cos \beta z \sin \omega t + \omega D \cos \beta z \cos \omega t - \omega C' \sin \beta z \sin \omega t + \omega D' \sin \beta z \cos \omega t \right]$$

$$H_y = \frac{\omega \epsilon_0}{\beta} \left[C \sin \beta z \sin \omega t - D \sin \beta z \cos \omega t - C' \cos \beta z \sin \omega t + D' \cos \beta z \cos \omega t \right]$$

Defining $\boxed{\eta_0 = \frac{\beta}{\omega\epsilon_0} = \frac{\omega\sqrt{\mu_0\epsilon_0}}{\omega\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}}}$, we have

$$H_y = \frac{1}{\eta_0} \left[C \sin \beta z \sin \omega t - D \sin \beta z \cos \omega t - C' \cos \beta z \sin \omega t + D' \cos \beta z \cos \omega t \right] \qquad \text{— (11)}$$

We now have to evaluate the arbitrary constants in (10) and (11). To do this, we recall that immediately adjacent to the current sheet, H_y has been found to be given by (5) and (6). Specifically, to the right of the sheet,

$$[H_y]_{z=0+} = \frac{J_{so}}{2} \cos \omega t$$

Using this with (11) gives us

$$\frac{1}{\eta_0} [-c' \sin \omega t + D' \cos \omega t] = \frac{J_{so}}{2} \cos \omega t$$

or $\quad c' = 0 \quad$ and $\quad D' = \dfrac{\eta_0 J_{so}}{2}$.

Thus

$$E_x = C \cos \beta z \cos \omega t + D \cos \beta z \sin \omega t + \frac{\eta_0 J_{so}}{2} \sin \beta z \sin \omega t$$

$$H_y = \frac{C}{\eta_0} \sin \beta z \sin \omega t - \frac{D}{\eta_0} \sin \beta z \cos \omega t + \frac{J_{so}}{2} \cos \beta z \cos \omega t$$

or

$$\boxed{\begin{aligned} E_x &= \left(\frac{C}{2} + \frac{\eta_0 J_{so}}{4}\right) \cos(\omega t - \beta z) + \left(\frac{C}{2} - \frac{\eta_0 J_{so}}{4}\right) \cos(\omega t + \beta z) \\ &\quad + \frac{D}{2} \sin(\omega t - \beta z) + \frac{D}{2} \sin(\omega t - \beta z) \qquad \text{for } z > 0 \\ H_y &= \left(\frac{C}{2\eta_0} + \frac{J_{so}}{4}\right) \cos(\omega t - \beta z) + \left(\frac{C}{2\eta_0} - \frac{J_{so}}{4}\right) \cos(\omega t + \beta z) \\ &\quad + \frac{D}{2\eta_0} \sin(\omega t - \beta z) - \frac{D}{2\eta_0} \sin(\omega t - \beta z) \qquad \text{for } z > 0 \end{aligned}}$$

Homework Problem (due 10/9/74)

We have considered the case of sinusoidal time variation of the fields and derived the solutions for E_x and H_y satisfying the two differential equations

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t}$$

For arbitrary time variation of the fields, show that the solutions for E_x and H_y are given by

$$E_x = A\, f(t - z/v) + B\, f(t + z/v)$$

$$H_y = \frac{1}{\eta_0}\left[A\, f(t - z/v) - B\, f(t + z/v)\right]$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ and $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

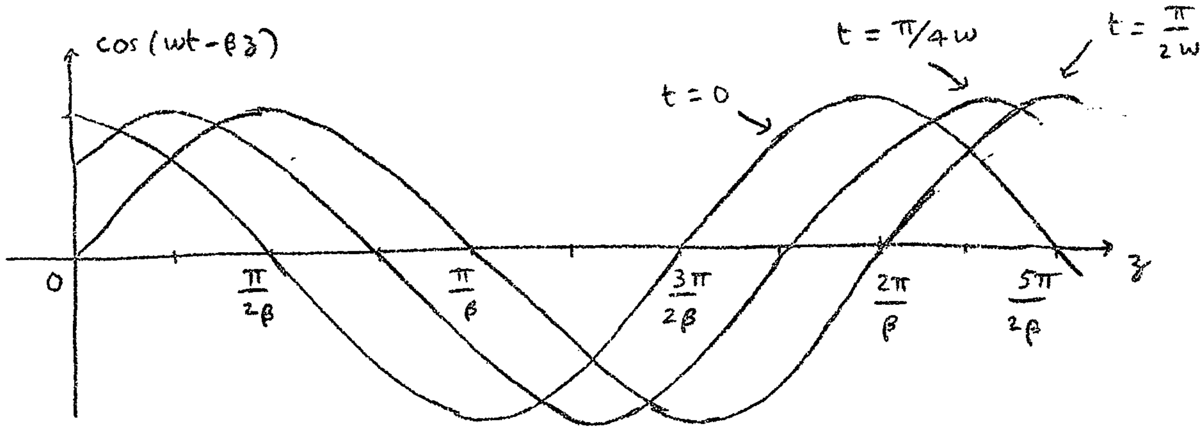
MEANING OF FUNCTIONS $\cos(\omega t \mp \beta z)$ AND $\sin(\omega t \mp \beta z)$

Let us consider $\cos(\omega t - \beta z)$.

For $t = 0$, $\cos(\omega t - \beta z) = \cos(-\beta z) = \cos \beta z$

For $t = \frac{\pi}{4\omega}$, $\cos(\omega t - \beta z) = \cos\left(\frac{\pi}{4} - \beta z\right)$

For $t = \frac{\pi}{2\omega}$, $\cos(\omega t - \beta z) = \cos\left(\frac{\pi}{2} - \beta z\right) = \sin \beta z$



We note from the above sketches that as time progresses, the function shifts bodily to the right, i.e. towards increasing values of z. Hence the function $\cos(\omega t - \beta z)$ represents a "traveling wave" progressing in the positive z direction. It is also known as the "positive going" wave or the "(+) wave".

We can find the velocity of propagation by dividing the distance moved by the time elapsed. Thus,

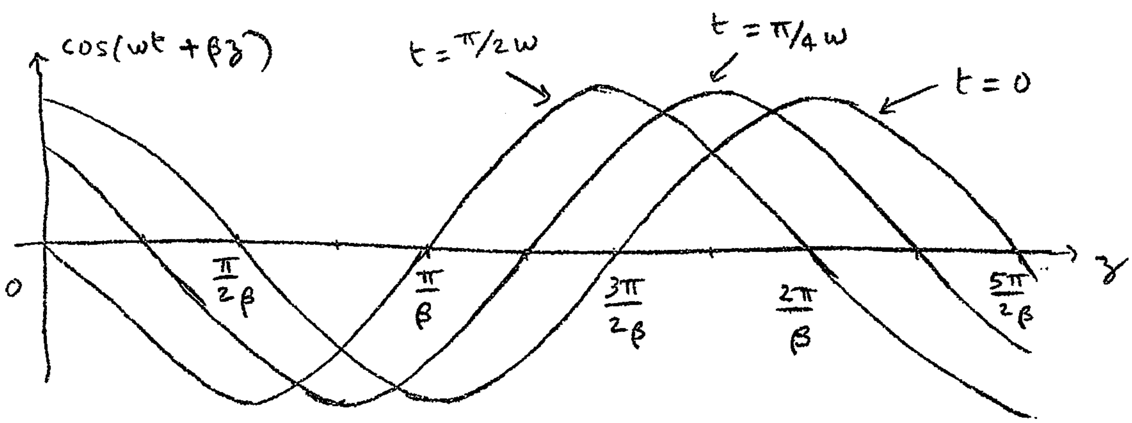
$$\text{velocity} = \frac{\pi/\beta - \pi/2\beta}{\pi/2\omega - 0} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/sec.}$$

Let us now consider the function $\cos(\omega t + \beta z)$.

For $t = 0$, $\cos(\omega t + \beta z) = \cos \beta z$

For $t = \frac{\pi}{4\omega}$, $\cos(\omega t + \beta z) = \cos\left(\frac{\pi}{4} + \beta z\right)$

For $t = \frac{\pi}{2\omega}$, $\cos(\omega t + \beta z) = \cos\left(\frac{\pi}{2} + \beta z\right) = -\sin \beta z$.



We note from the above sketches that as time progresses, the function shifts bodily to the left, i.e., towards the decreasing values of z. Hence the function $\cos(\omega t + \beta z)$ represents a "<u>traveling wave</u>" progressing in the negative z direction. It is also known as the "<u>negative going</u>" wave or the "<u>(−) wave</u>." We can once again find the velocity of propagation by dividing the distance moved by the time elapsed. Thus,

$$\text{velocity} = \frac{\pi/\beta - 3\pi/2\beta}{\pi/2\omega - 0} = -\frac{\omega}{\beta} = -\frac{\omega}{\omega\sqrt{\mu_0\epsilon_0}} = -\frac{1}{\sqrt{\mu_0\epsilon_0}} = -3\times10^{8}\ \text{m/sec}$$

The (−) sign signifies travel in the negative z direction.

To generalize the discussion on the previous two pages, let us consider an arbitrary function f of $\left(t - \frac{\beta}{\omega} z\right)$, i.e., $f\left(t - \frac{\beta}{\omega} z\right)$. We note that as time progresses, one has to move in the z direction in order to view the same value of f always. The velocity with which the observer has to move can be obtained by considering two pairs of values of t and z. Thus if we consider (t_1, z_1) and (t_2, z_2), the values of the function for these two pairs are $f\left(t_1 - \frac{\beta}{\omega} z_1\right)$ and $f\left(t_2 - \frac{\beta}{\omega} z_2\right)$. In order for the observer to see the same value of f, the arguments of f must be equal, i.e.,

$$t_1 - \frac{\beta}{\omega} z_1 = t_2 - \frac{\beta}{\omega} z_2$$

or

$$(z_2 - z_1)\frac{\beta}{\omega} = (t_2 - t_1)$$

$$\frac{z_2 - z_1}{t_2 - t_1} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{1}{\sqrt{\mu_0\epsilon_0}}.$$

Thus, $f\left(t - \frac{\beta}{\omega} z\right) = f\left(t - \sqrt{\mu_0\epsilon_0}\, z\right)$ represents a "traveling wave" propagating in the positive z direction with velocity $1/\sqrt{\mu_0\epsilon_0}$. Likewise, it can be shown that any arbitrary function g of $\left(t + \frac{\beta}{\omega} z\right)$, i.e., $g\left(t + \frac{\beta}{\omega} z\right) = g\left(t + \sqrt{\mu_0\epsilon_0}\, z\right)$ represents a "(−) wave," i.e., a wave propagating in the negative z direction, with velocity $1/\sqrt{\mu_0\epsilon_0}$.

<u>Examples</u>: $\cos\omega\left(t - \frac{\beta}{\omega} z\right) = \cos(\omega t - \beta z)$

$$e^{(t - \sqrt{\mu_0\epsilon_0}\, z)}, \quad (t + \sqrt{\mu_0\epsilon_0}\, z)^2, \quad e^{-(t - \sqrt{\mu_0\epsilon_0}\, z)} \sin\omega(t - \sqrt{\mu_0\epsilon_0}\, z)$$

Returning now to the problem of the infinite plane sheet of current for which we wish to find the electromagnetic field, we recall that the solutions for E_x and H_y were obtained to be

$$E_x(z,t) = \frac{2C + \eta_0 J_{s0}}{4}\cos(\omega t - \beta z) + \frac{2C - \eta_0 J_{s0}}{4}\cos(\omega t + \beta z)$$
$$+ \frac{D}{2}\sin(\omega t - \beta z) + \frac{D}{2}\sin(\omega t + \beta z) \quad \text{for } z > 0$$

$$H_y(z,t) = \frac{2C + \eta_0 J_{s0}}{4\eta_0}\cos(\omega t - \beta z) - \frac{2C - \eta_0 J_{s0}}{4\eta_0}\cos(\omega t + \beta z)$$
$$+ \frac{D}{2\eta_0}\sin(\omega t - \beta z) - \frac{D}{2\eta_0}\sin(\omega t + \beta z) \quad \text{for } z > 0$$

We now recognize that these solutions represent superpositions of (+) and (−) traveling waves, i.e., waves moving away and waves moving towards the current sheet. In the region $z > 0$, we however have to rule out (−) waves in the absence of a source or a reflecting object to the right of the sheet. Thus for $z > 0$,

$$2C - \eta_0 J_{s0} = 0 \quad \text{or} \quad C = \frac{\eta_0 J_{s0}}{2}$$
$$D = 0$$

The solutions then reduce to

$$\left.\begin{aligned} E_x(z,t) &= \frac{\eta_0 J_{s0}}{2}\cos(\omega t - \beta z) \\ H_y(z,t) &= \frac{J_{s0}}{2}\cos(\omega t - \beta z) \end{aligned}\right\} \text{for } z > 0$$

Likewise, we can find the solutions for the fields to the left of the current sheet, i.e., in the region $z<0$. In fact, we can immediately write down that

$$H_y(z,t) = -\frac{J_{s0}}{2}\cos(\omega t+\beta z) \quad \text{for } z<0$$

since we know that it must correspond to a (-) wave and it must reduce to $-\frac{J_{s0}}{2}\cos\omega t$ for $z=0-$. Having done this, we can then use one of the Maxwell's equations and find E_x. Thus

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu_0\frac{J_{s0}}{2}\omega\sin(\omega t+\beta z)$$

$$E_x = \frac{\mu_0 J_{s0}}{2}\frac{\omega}{\beta}\cos(\omega t+\beta z).$$

$$\mu_0\frac{\omega}{\beta} = \frac{\mu_0}{\sqrt{\mu_0\epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}.$$

$$= \frac{\eta_0 J_{s0}}{2}\cos(\omega t+\beta z) \quad \text{for } z<0.$$

Thus we finally have

$$\boxed{\begin{aligned} E_x &= \frac{\eta_0 J_{s0}}{2}\cos(\omega t\mp\beta z) \quad \text{for } z\gtrless 0 \\ H_y &= \pm\frac{J_{s0}}{2}\cos(\omega t\mp\beta z) \quad \text{for } z\gtrless 0 \end{aligned}}$$

These solutions are said to correspond to "uniform plane waves," since for any fixed time, the phase of the fields, i.e., the quantity $(\omega t\mp\beta z)$ is uniform over the planes $z=$ constant.

The magnitude of the rate of change of phase with distance for any fixed time $t_0 = \left| \frac{d}{d} (\omega t_0 \mp \beta z) \right| = \beta$.

Hence β is known as the phase constant. Units are radians/m.

The velocity of propagation is known as the phase velocity, v_p because it is the velocity with which a given constant phase surface (z = constant) moves along the direction of propagation.

The distance in which the phase changes by 2π radians at fixed time is known as the wavelength, λ. Thus

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

From $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/v_p$, we then have

$$\lambda = \frac{2\pi}{\omega/v_p} = \frac{v_p}{\omega/2\pi} = \frac{v_p}{f}$$

or $$\boxed{\lambda f = v_p}$$

For free space

$$\lambda \text{ in meters} \times f \text{ in Hz} = 3 \times 10^8$$

$$\boxed{\lambda \text{ in meters} \times f \text{ in MHz} = 3 \times 10^2 = 300}$$

$$\boxed{\frac{E_x}{H_y} = \pm \eta_0 \quad \text{for } z \gtrless 0\text{, i.e. } \mp \text{ wave}}$$

$\eta_0 = \sqrt{\mu_0/\epsilon_0} = 120\pi\ \Omega = 377\Omega$ is known as the intrinsic impedance.

EXAMPLE: $E_x = 10 \cos(3\pi \times 10^8 t - \pi z)$

Radian frequency, $\omega = 3\pi \times 10^8$ radians/sec.

frequency, $f = \dfrac{\omega}{2\pi} = \dfrac{3\pi \times 10^8}{2\pi} = 1.5 \times 10^8 \text{ Hz} = 150 \text{ MHz}$

Phase constant, $\beta = \pi$ radians/m

Wavelength, $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{\pi} = 2$ meters.

Phase velocity, $v_p = \lambda f = 2 \times 1.5 \times 10^8 = 3 \times 10^8$ m/sec.

Also note that $v_p = \dfrac{\omega}{\beta} = \dfrac{3\pi \times 10^8}{\pi} = 3 \times 10^8$ m/sec.

Since the field corresponds to a (+) wave,

$$H_y = \frac{E_x}{\eta_0} = \frac{10}{377} \cos(3\pi \times 10^8 t - \pi z).$$

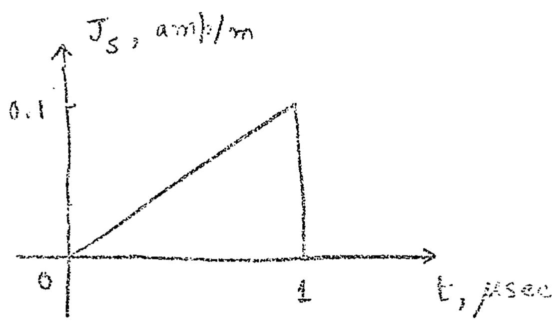
HOMEWORK PROBLEM (due 10/11/74)

Assume that the time variation of the current density of the infinite plane current sheet, instead of being sinusoidal, is given as shown in the accompanying figure. Sketch

(a) E_x vs. t at $z = 300$ meters

(b) E_x vs. z for $t = 1$ μsec.

(c) H_y vs. z for $t = 2$ μsec.



DOPPLER EFFECT:

We found that an infinite plane sheet of current with density

$$\vec{J}_s = -J_{s0} \cos \omega t \; \vec{i}_x \qquad \text{on } z = 0$$

produces uniform plane electromagnetic waves given by

$$\vec{E} = \frac{\eta_0 J_{s0}}{2} \cos(\omega t \mp \beta z) \qquad \text{for } z \geq 0$$

$$\vec{H} = \pm \frac{J_{s0}}{2} \cos(\omega t \mp \beta z) \qquad \text{for } z \leq 0$$

For a fixed observer at $z = z_0$ and considering the positive going wave, the phase of the field is given by

$$\phi_{fixed} = \omega t - \beta z_0$$

$$\frac{d\phi_{fixed}}{dt} = \omega$$

Thus the frequency viewed by a fixed observer is the same as the frequency of the source of the wave.

On the other hand, consider an observer moving in the positive z direction with a velocity v_0 meters/sec and starting at $z = z_0$ for $t = 0$, i.e., the position of the observer is given by

$$z = z_0 + v_0 t$$

and the phase of the wave as viewed by the moving observer is

$$\phi_{moving} = \omega t - \beta(z_0 + v_0 t)$$

$$= (\omega - \beta v_0) t - \beta z_0$$

$$\frac{d\phi_{moving}}{dt} = \omega - \beta v_0 = \omega\left(1 - \frac{\beta}{\omega} v_0\right)$$

$$= \omega\left(1 - \frac{v_0}{v_p}\right)$$

$$\boxed{\frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = v_p}$$

Thus the frequency viewed by the moving observer is not the same as the source frequency ω. This phenomenon is known as the doppler shift. The doppler shifted frequency is less than the actual frequency.

The amount of doppler shift in the radian frequency $= \omega \dfrac{v_0}{v_p}$

The amount of doppler shift in the linear frequency $= f \dfrac{v_0}{v_p}$

or $\boxed{\Delta f_D = \dfrac{v_0}{\lambda}}$ since $v_p = \lambda f$

For an observer moving with the wave, the doppler shifted frequency is less than the actual frequency by the amount $\dfrac{v_0}{\lambda}$ and for an observer moving opposite to the wave, the doppler shifted frequency is greater than the actual frequency by the amount $\dfrac{v_0}{\lambda}$.

<u>EXAMPLE</u>: For doppler radar frequency $f = 5$ GHz $= 5{,}000$ MHz,

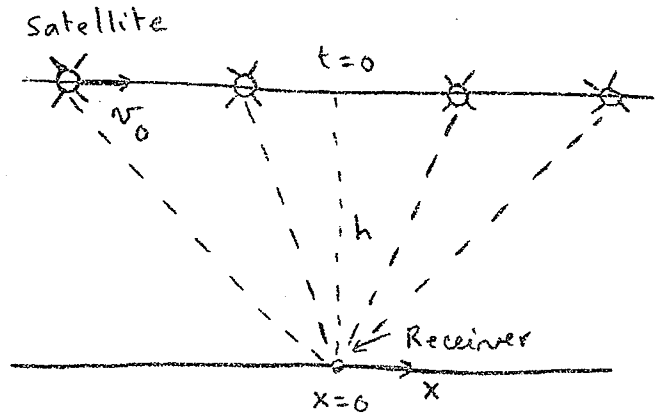
$$\lambda = \frac{300}{5000} \text{ m}$$

For a car traveling at 100 km/hr ($\approx$ 60 miles/hr), i.e., for

$$v_0 = \frac{100 \times 10^3}{3600} \text{ m/sec}$$

$$\Delta f_D = \frac{100 \times 10^3}{3600} \times \frac{5000}{300} = 463 \text{ Hz}.$$

Let us now consider the case of a satellite. We shall assume plane earth for simplicity.



Let the satellite be overhead at $t=0$ and its velocity be v_0 as shown in the figure. The distance between the satellite and the receiver is then equal to $\sqrt{h^2+v_0^2t^2}$. The phase of the wave observed at the receiver is therefore given by

$$\phi_{obs} = \omega t - \beta\sqrt{h^2+v_0^2t^2}$$

The doppler shifted frequency observed at the receiver is

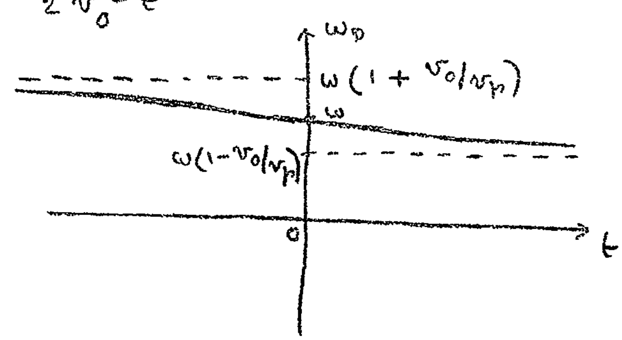
$$\omega_D = \frac{d\phi_{obs}}{dt} = \frac{d}{dt}\left[\omega t - \beta\sqrt{h^2+v_0^2t^2}\right]$$

$$= \omega - \beta\cdot\frac{1}{2}\left(h^2+v_0^2t^2\right)^{-1/2} 2v_0^2t$$

$$= \omega - \frac{\beta v_0^2 t}{\sqrt{h^2+v_0^2t^2}}$$

$$= \omega - \frac{\omega}{v_p}\frac{v_0^2t}{\sqrt{h^2+v_0^2t^2}}$$

$$= \omega\left[1 - \frac{v_0}{v_p}\frac{v_0t}{\sqrt{h^2+v_0^2t^2}}\right]$$



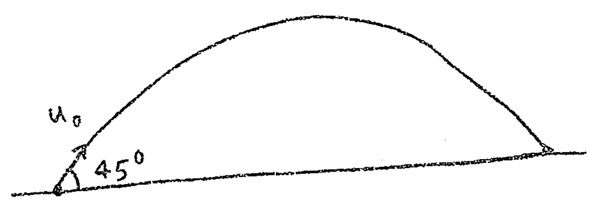
EXAMPLE: For a satellite employing $f = 40$ MHz and travelling at $v_0 = 7$ km/sec,

$$\text{Maximum doppler shift} = 40 \times 10^6 \times \frac{7 \times 10^3}{3 \times 10^8}$$

$$= 933 \text{ Hz.}$$

HOMEWORK PROBLEM (Due 10/18/74)

An experimental rocket is fired with an initial velocity u_0 meters/sec and making an angle of 45° with the horizontal. Communication is maintained with the rocket from a transmitter at the rocket launching site having a frequency f. Show that the received frequency when the rocket is at its apogee is doppler shifted by the amount $\frac{f}{c}(0.8944\, u_0)$. Assume plane earth and free space. c is the velocity of light in free space.



POYNTING VECTOR AND ENERGY STORAGE

We found that the solution for the electromagnetic field due to the infinite plane current sheet characterized by

$\vec{J}_s = -J_{so} \cos \omega t \; \vec{i}_x$

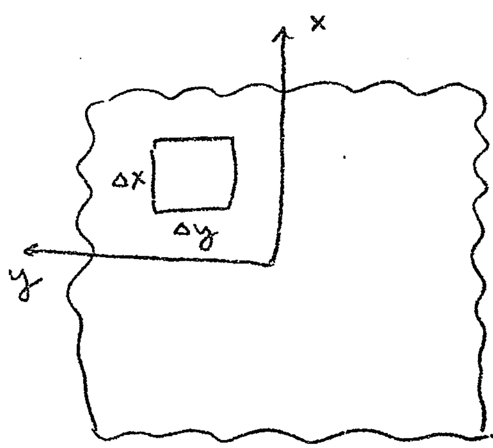
is given by

$$E_x = \frac{\eta_0 J_{so}}{2} \cos(\omega t \mp \beta z) \quad \text{for } z \gtrless 0$$

$$H_y = \pm \frac{J_{so}}{2} \cos(\omega t \mp \beta z) \quad \text{for } z \gtrless 0$$

Since the current is flowing against the electric field on the sheet, certain amount of work must be done by the source of the current in order to maintain the current flow on the sheet.

Let us consider a rectangular area of length Δx and width Δy on the current sheet.



Since the current density is $J_{so} \cos \omega t$, the current crossing the width Δy

$= J_{so} \cos \omega t \cdot \Delta y$ amp.

$\therefore$ The charge crossing the width Δy in time dt is given by

$dq = J_{so} \, \Delta y \, \cos \omega t \cdot dt$ coulombs

The force exerted on this charge by the electric field on the sheet is

$$\vec{F} = dq\,\vec{E} = J_{so}\,\Delta y\,\cos\omega t\,dt \cdot E_x\,\vec{i}_x$$

The amount of work required to be done against the electric field in displacing this charge by the distance Δx is

$$dW = F_x\,\Delta x = J_{so}\,\Delta y\,\cos\omega t\,dt\;E_x\,\Delta x$$

$$= J_{so}\,E_x\,\cos\omega t\,dt\;\Delta x\,\Delta y$$

The power supplied by the source of the current in maintaining the surface current over the area $\Delta x\,\Delta y$ is

$$\frac{dW}{dt} = J_{so}\,E_x\,\cos\omega t\;\Delta x\,\Delta y$$

$$= \eta_0\,\frac{J_{so}^2}{2}\,\cos^2\omega t\;\Delta x\,\Delta y$$

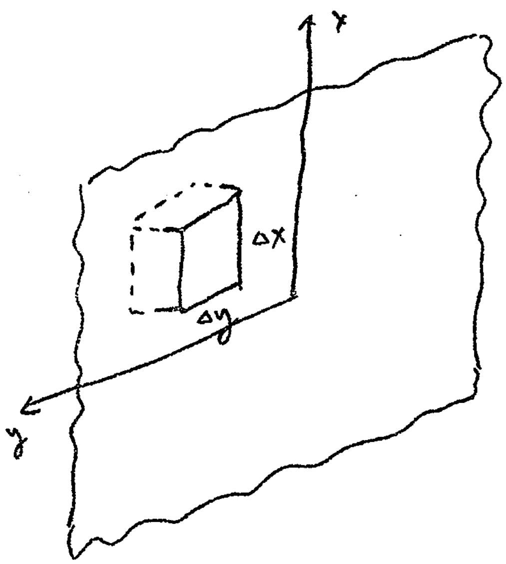
We would expect this power to be carried by the electromagnetic wave. To investigate this, we note that the quantity $\vec{E}\times\vec{H}$ has the units of

$$\frac{\text{Newtons}}{\text{Coulomb}} \times \frac{\text{Amperes}}{\text{Meter}} = \frac{\text{Newtons}}{\text{Coulomb}} \times \frac{\text{Coulomb}}{\text{Sec - meter}} \times \frac{\text{Meter}}{\text{Meter}}$$

$$= \frac{\text{Newton - meters}}{\text{second}} \times \frac{1}{(\text{Meter})^2} = \frac{\text{Watts}}{(\text{Meter})^2}$$

which represents power density.

Let us then consider a rectangular box as shown.



Evaluating the surface integral of $\vec{E} \times \vec{H}$ over the surface of the box, we obtain the power flow out of the box as

$$\oint \vec{E} \times \vec{H} \cdot \vec{ds}$$

$$= \frac{\eta_0 J_{s0}^2}{4} \cos^2 \omega t \; \vec{i}_z \cdot \Delta x \, \Delta y \, \vec{i}_z + \left(-\frac{\eta_0 J_{s0}^2}{4} \cos^2 \omega t \; \vec{i}_z\right) \cdot \left(-\Delta x \, \Delta y \, \vec{i}_z\right)$$

$$= \frac{\eta_0 J_{s0}^2}{2} \cos^2 \omega t \; \Delta x \, \Delta y$$

which is exactly the same as the power supplied by the source of the current as obtained on the previous page.

Thus we interpret the quantity $\vec{E} \times \vec{H}$ as the power flow density vector associated with the electromagnetic field. It is known as the <u>Poynting vector.</u>

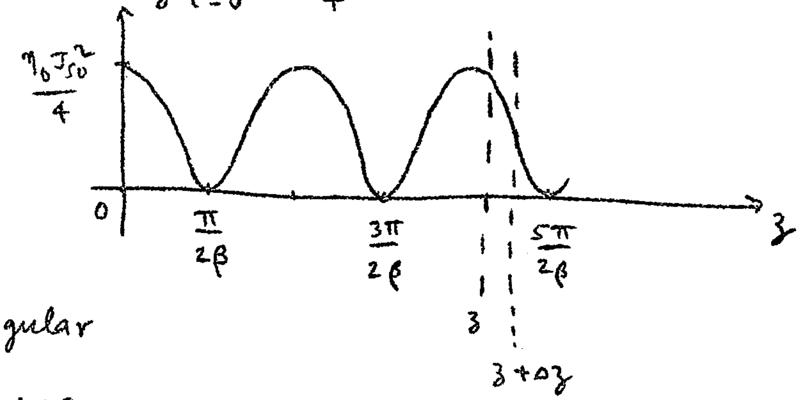
Let us now consider the Poynting vector in the region $z > 0$, i.e.,

$$\vec{P} = \vec{E} \times \vec{H} = E_x H_y \vec{i}_z$$

$$= \frac{\eta_0 J_{s0}^2}{4} \cos^2(\omega t - \beta z)\, \vec{i}_z$$

$$[P_z]_{t=0} = \frac{\eta_0 J_{s0}^2}{4} \cos^2 \beta z$$

The variation of P_z with z for $t = 0$ is shown in the figure.



If we now consider a rectangular box lying between z and $z + \Delta z$ and having dimensions Δx and Δy, and evaluate the power flow out of the box, we get

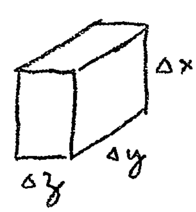
$$\oint_S \vec{P} \cdot d\vec{s} = [P_z]_{z+\Delta z} \Delta x \Delta y - [P_z]_z \Delta x \Delta y$$

$$= \frac{[P_z]_{z+\Delta z} - [P_z]_z}{\Delta z} \cdot \Delta x \Delta y \Delta z$$

$$= \frac{\partial P_z}{\partial z} \Delta x \Delta y \Delta z = \frac{\partial P_z}{\partial z} \Delta v$$

$$= \frac{\partial}{\partial z} [E_x H_y] \Delta v$$

$$= \left[\frac{\partial E_x}{\partial z} H_y + E_x \frac{\partial H_y}{\partial z} \right] \Delta v$$



where $\Delta v = \Delta x \Delta y \Delta z$ is the volume of the box.

But $\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$ and $\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t}$. Hence

$$\oint_S \vec{P} \cdot d\vec{s} = \left[-\frac{\partial B_y}{\partial t} \cdot H_y - E_x \frac{\partial D_x}{\partial t} \right] \Delta v$$

$$= \left[-\mu_0 H_y \frac{\partial H_y}{\partial t} - \epsilon_0 E_x \frac{\partial E_x}{\partial t} \right] \Delta v$$

$$= \left[-\frac{\partial}{\partial t} \left(\frac{1}{2} \mu_0 H_y^2 \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E_x^2 \right) \right] \Delta v$$

$$= -\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E_x^2 \Delta v \right) - \frac{\partial}{\partial t} \left(\frac{1}{2} \mu_0 H_y^2 \Delta v \right)$$

Thus the power flow out of the box is equal to the time rate of decrease of the quantity $\frac{1}{2} \epsilon_0 E_x^2 \Delta v$ plus the time rate of decrease of the quantity $\frac{1}{2} \mu_0 H_y^2 \Delta v$. The quantities $\frac{1}{2} \epsilon_0 E_x^2$ and $\frac{1}{2} \mu_0 H_y^2$ are the stored energy densities associated with the electric and magnetic fields, respectively.

HOMEWORK PROBLEM (Due 10/21/74)

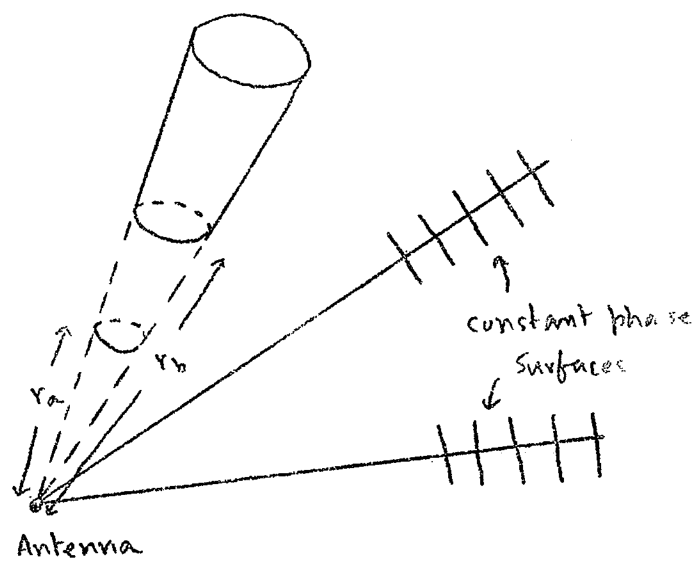
Show that the quantities $\frac{1}{2} \epsilon_0 E^2$ and $\frac{1}{2} \mu_0 H^2$ have the units joules / m^3.

SOME EXAMPLES

EXAMPLE 1. Radiation Field of an Antenna

Radiation field is the field existing at several wavelengths away from a physical antenna. At such distances from the antenna, the radiated waves are approximately uniform plane waves with their constant phase surfaces normal to the radial lines drawn from the antenna.

The Poynting vector $\vec{E} \times \vec{H}$ is directed everywhere in the radial direction with $\vec{E}$ and $\vec{H}$ lying in the planes of constant phase and perpendicular to each other. Thus the Poynting vector is proportional to the square of the magnitude of the electric field intensity.



Now if we consider two spherical surfaces of radii r_a and r_b centered at the antenna and insert a cone through these surfaces such that its vertex is at the antenna, the power crossing the portion of the spherical surface of radius r_a must be the same as the power crossing the portion of the spherical surface of radius r_b. Since these surface areas are proportional to the square of the radius and since the surface integral of the Poynting vector gives the power, it follows that the Poynting vector must be inversely proportional to the square of the radial distance. Thus the electric and magnetic field intensities are inversely proportional to the radial distance r.

$$\vec{E} \sim \frac{1}{r} \quad \text{and} \quad \vec{H} \sim \frac{1}{r}$$

$$\vec{P} \sim \frac{1}{r^2}$$

As a numerical example, in earth to moon communication, the reduction in the field intensities due to the $\frac{1}{r}$ factor is by a factor of $\frac{10^{-7}}{38}$ (Distance from earth to moon = 38×10^4 km = 38×10^7 m)

In terms of decibel notation, the reduction

$$= 20 \log_{10} 38 \times 10^7$$

$$= 20 \log_{10} 38 + 20 \log_{10} 10^7$$

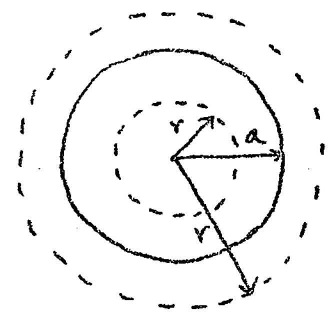
$$= 20 \times 1.58 + 20 \times 7$$

$$= 31.6 + 140$$

$$= 171.6 \text{ db.}$$

EXAMPLE 2. Work required to assemble a spherical ball of charge radius a and uniform charge density ρ_0 C/m³.

This work is obviously the same as the energy stored in the electric field of the charge distribution. In the Homework Problem due on 9/23/74, the electric field due to the above charge distribution was computed by using Gauss' law in integral form. We shall repeat the solution here briefly.



Choosing a Gaussian surface of radius r, we have

$$4\pi r^2 D_r = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & \text{for } r < a \\ \frac{4}{3}\pi a^3 \rho_0 & \text{for } r > a \end{cases}$$

$$D_r = \begin{cases} \frac{\rho_0 r}{3} & \text{for } r < a \\ \frac{\rho_0 a^3}{3 r^2} & \text{for } r > a \end{cases}$$

$$E_r = \begin{cases} \frac{\rho_0 r}{3\epsilon_0} & \text{for } r < a \\ \frac{\rho_0 a^3}{3\epsilon_0 r^2} & \text{for } r > a \end{cases}$$

Then the energy density stored in the electric field of the spherical ball of charge is given by

$$W_e = \frac{1}{2}\epsilon_0 E_r^2 = \begin{cases} \dfrac{\rho_0^2 r^2}{18\,\epsilon_0} & \text{for } r < a \\[2ex] \dfrac{\rho_0^2 a^6}{18\,\epsilon_0 r^4} & \text{for } r > a \end{cases}$$

The total energy stored in the electric field

$$= \int W_e \, dv$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\rho_0^2 r^2}{18\epsilon_0} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$+ \int_{r=a}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\rho_0^2 a^6}{18\,\epsilon_0 r^4} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{4\pi\rho_0^2}{18\,\epsilon_0}\left[\frac{r^5}{5}\Big|_0^a - \frac{a^6}{r}\Big|_a^\infty \right]$$

$$= \frac{4\pi\rho_0^2}{18\epsilon_0}\left(\frac{a^5}{5} + a^5\right) = \frac{4\pi\rho_0^2}{18\epsilon_0} \times \frac{6a^5}{5}$$

$$= \frac{4\pi\rho_0^2 a^5}{15\,\epsilon_0} \text{ joules.}$$

HOMEWORK PROBLEM (Due 10/23/74)

A charge Q coulombs is distributed uniformly in the volume of a sphere of radius a meters. It is proposed to extract energy by dividing the spherical charge into two uniformly distributed spherical charges, each of value $\frac{Q}{2}$ coulombs and by slowly separating them infinitely apart. Two choices are available for forming the two spherical charges from the original spherical charge:

(a) by keeping the radii the same as the radius of the original charge and reducing the densities;

(b) by keeping the densities the same as the density of the original charge and decreasing the radii.

Three persons A, B and C enter into an argument. A contends that more energy can be extracted by resorting to choice (a) than by following choice (b). B is of just the opposite view. C maintains that the same amount of energy can be extracted irrespective of whether choice (a) or choice (b) is used.

Determine which one of three persons is correct by evaluating the energy which can be extracted in each of the two cases.

MATERIALS

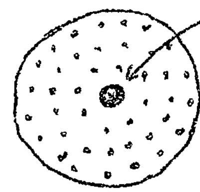
can be classified into three types:

(a) Conductors — conduction

(b) Dielectrics — Polarization

(c) Magnetic materials — Magnetization

CONDUCTORS



classical model of an atom

The electrons orbiting and spinning around the nucleus can be divided into two types

a) bound electrons

b) free or conduction electrons

Bound electrons can only be displaced but not removed from the influence of the nucleus whereas conduction electrons are constantly under thermal agitation being released from one parent atom at one point and recaptured by another atom at a different point.

In the absence of an applied field, the motion of the conduction electrons is completly random, i.e., the average thermal velocity on a "macroscopic" scale is zero.

With the application of an electric field, an additional velocity is superimposed on the random velocities. Due to the frictional mechanism provided by collisions of the electrons with the atomic lattice, the electrons instead of accelerating under the influence of the applied field drift with an average velocity proportional in magnitude to the applied field. This process is known as conduction.

Materials in which conduction is the predominant process are known as <u>conductors</u>. The drift velocity is given by

$$\vec{v}_d = -\mu_e \vec{E}$$

where μ_e is known as the mobility of the electron. If the number of free electrons participating in conduction is N_e per cubic meter of the material, then the conduction current density is given by

$$\vec{J}_c = N_e e \vec{v}_d = -\mu_e N_e e \vec{E}$$

$$= \mu_e N_e |e| \vec{E}$$

we define σ, known as the conductivity of the material as

$$\sigma = \mu_e N_e |e| \quad \text{mhos/m}$$

Thus, for conductors,

$$\boxed{\vec{J}_c = \sigma \vec{E}}$$

For example, conductivities of a few materials are given below:

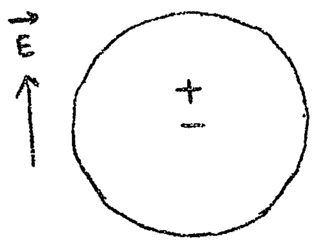
Copper	5.8×10^7 ℧/m	Sea water	4
Aluminum	3.5×10^7 ℧/m	Fresh water	10^{-3}
Lead	4.8×10^6 ℧/m	Fused quartz	0.4×10^{-17}

In considering electromagnetic wave propagation in conducting media, the conduction current density must be employed for the current density term on the right side of Ampere's circuital law.

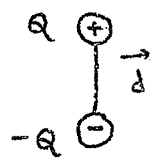
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}.$$

DIELECTRICS

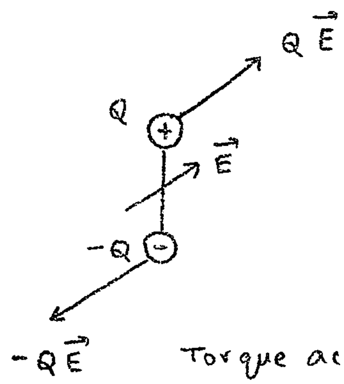
Dielectrics are characterized by polarization. Polarization is the phenomenon by means of which the bound electrons of an atom are displaced such that the centroid of the electron cloud is separated from the centroid of the nucleus.



Atomic polarization



Electric dipole



Torque acting on a dipole gives rise to orientational polarization

The strength of an electric dipole created due to polarization in the material is defined by the electric dipole moment $\vec{p}$ given by

$$\vec{p} = Q\vec{d} \qquad \text{Coulomb-meter}$$

Note that $\vec{p}$ is proportional to Q and the separation $\vec{d}$.

On a macroscopic scale, we define

$$\vec{P} = \frac{1}{\Delta v}\sum_{j=1}^{N\Delta v} \vec{p}_j = N\vec{p} \qquad \frac{\text{coulomb-meter}}{\text{meter}^3} \text{ or } \frac{\text{Coulomb}}{\text{meter}^2}$$

where N is the number of molecules per unit volume of the material, and $\vec{p}$ is the average dipole moment per molecule. $\vec{P}$ is known as the "polarization vector" or the electric dipole moment per unit volume.

For many dielectric materials, $\vec{P}$ is related to $\vec{E}$ in the simple manner

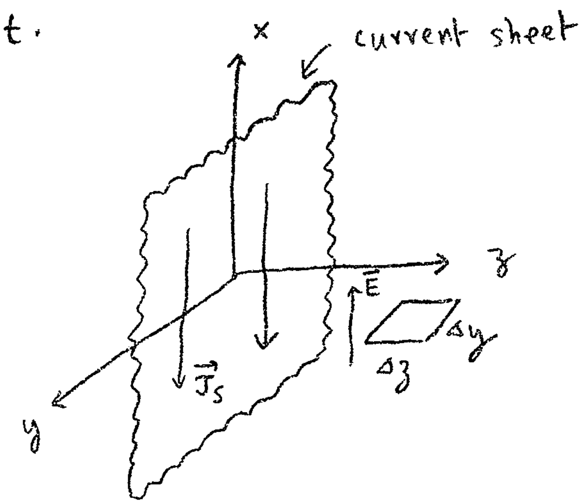
$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where χ_e, a dimensionless parameter known as the "electric susceptibility" is a measure of the ability of the material to become polarized and differs from one dielectric to another.

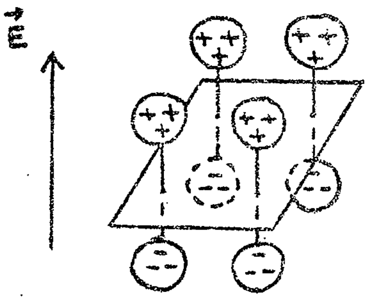
Let us consider the infinite plane current sheet with a dielectric medium on either side of the sheet instead of free space. Then the electric field induces polarization in the dielectric as given by

$$\vec{P} = P_x \vec{i}_x = \epsilon_0 \chi_e E_x \vec{i}_x$$

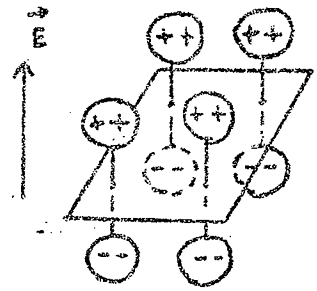
Note that E_x is a function of z and t.



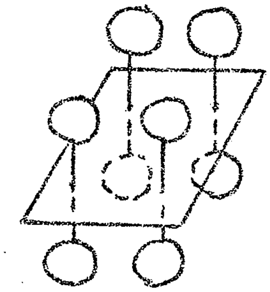
If we consider an infinitesimal area $\Delta y \Delta z$ at a fixed point as shown in the figure, then the electric field at that point varies in a sinusoidal manner with time, i.e., $E_0 \cos \omega t$ and the induced dipole moments vary accordingly as shown on the following page.



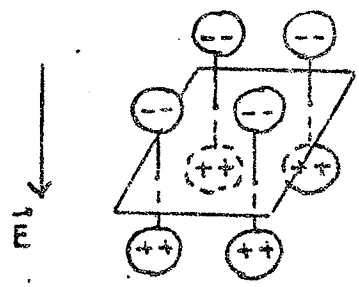
$\omega t = 0$



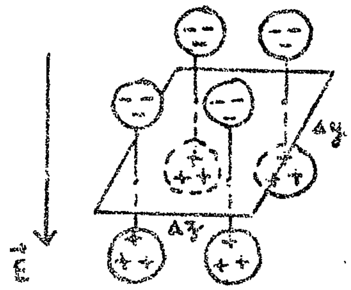
$\omega t = \pi/4$



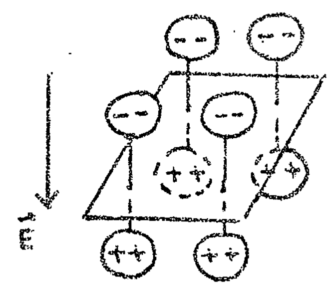
$\omega t = \pi/2$



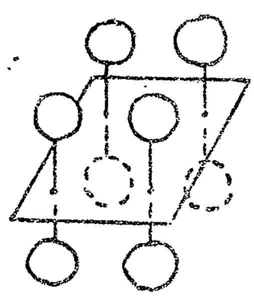
$\omega t = 3\pi/4$



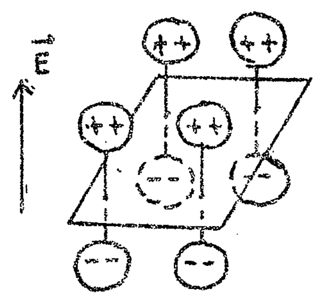
$\omega t = \pi$



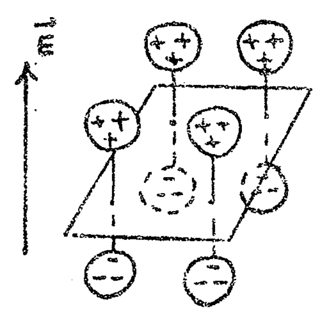
$\omega t = 5\pi/4$



$\omega t = 3\pi/2$



$\omega t = 7\pi/4$



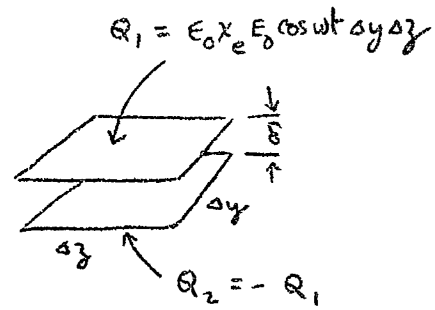
$\omega t = 2\pi$

The situation can be considered as two plane sheets of equal and opposite time varying charges displaced by an amount δ in the x direction. To find the magnitude of either charge, we note that the total dipole moment in the volume $\delta \Delta y \Delta z$

$$= P_x (\delta \Delta y \Delta z) = \epsilon_0 \chi_e E_0 \cos \omega t \; \delta \Delta y \Delta z$$

$$= (\epsilon_0 \chi_e E_0 \cos \omega t \; \Delta y \Delta z) \, \delta$$

$$= Q_1 \delta$$



The time varying charges are equivalent to a current flowing across the surface $\Delta y \Delta z$. This current, known as the "polarization current," is given by

$$I_{px} = \frac{dQ_1}{dt} = -\frac{dQ_2}{dt} = -\epsilon_0 \chi_e E_0 \omega \sin \omega t \; \Delta y \Delta z$$

or the polarization current density is given by

$$J_{px} = \frac{I_{px}}{\Delta y \Delta z} = -\epsilon_0 \chi_e E_0 \omega \sin \omega t$$

$$= \frac{\partial}{\partial t} (\epsilon_0 \chi_e E_0 \cos \omega t) = \frac{\partial P_x}{\partial t}$$

or

$$\boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}}$$

In considering electromagnetic wave propagation in a dielectric medium, the polarization current density must be employed for the current density term on the right side of Ampere's circuital law.

$$\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_p + \frac{\partial}{\partial t}(\epsilon_0 \vec{E})$$
$$= \vec{J} + \frac{\partial \vec{P}}{\partial t} + \frac{\partial}{\partial t}(\epsilon_0 \vec{E})$$
$$= \vec{J} + \frac{\partial}{\partial t}(\epsilon_0 \vec{E} + \vec{P})$$

($\vec{J}$ represents any other current density which may be present in addition to the polarization current density)

We now <u>redefine</u> $\vec{D}$ to read as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$= \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$
$$= \epsilon_0 (1 + \chi_e) \vec{E}$$
$$= \epsilon_0 \epsilon_r \vec{E}$$ where ϵ_r is the relative permittivity $(= 1 + \chi_e)$
$$\boxed{\vec{D} = \epsilon \vec{E}}$$ where ϵ is the permittivity $(= \epsilon_0 \epsilon_r)$

so that by replacing ϵ_0 by ϵ, we can <u>avoid</u> taking into account <u>explicitly</u> the polarization current density! The quantity ϵ takes into account the effects of polarization and we can forget about them.

The relative permittivities for a few materials are given below.

Air	1.0006	Dry Earth	5
Teflon	2.1	Mica	6
Polystyrene	2.56	Wet Earth	10
Fused quartz	3.8	Distilled water	81
Bakelite	4.9	Titanium dioxide	100

EXAMPLE: Certain types of materials are anisotropic, that is, their properties differ for different directions of the applied fields. For example for anisotropic dielectrics, each component of the polarization vector can in general be dependent on all components of the electric field intensity. This in turn results in each component of $\vec{D}$ to be in general dependent on all components of $\vec{E}$. For example, a particular anisotropic dielectric may be characterized by

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 7\epsilon_0 & 2\epsilon_0 & 0 \\ 2\epsilon_0 & 4\epsilon_0 & 0 \\ 0 & 0 & 3\epsilon_0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

For $\vec{E} = E_0 \vec{i}_z$, $\vec{D} = 3\epsilon_0 E_0 \vec{i}_z$, $\vec{D} \parallel \vec{E}$

$\vec{E} = E_0 \vec{i}_x$, $\vec{D} = 7\epsilon_0 E_0 \vec{i}_x + 2\epsilon_0 E_0 \vec{i}_y$, $\vec{D}$ not $\parallel \vec{E}$

$\vec{E} = E_0 \vec{i}_y$, $\vec{D} = 2\epsilon_0 E_0 \vec{i}_x + 4\epsilon_0 E_0 \vec{i}_y$, $\vec{D}$ not $\parallel \vec{E}$

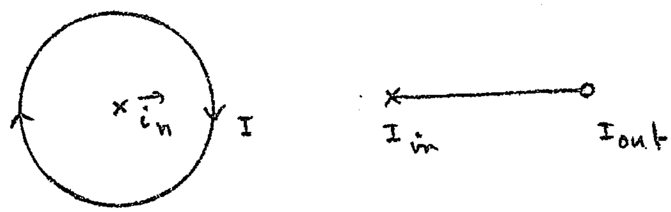
$\vec{E} = E_0(\vec{i}_x + 2\vec{i}_y)$, $\vec{D} = 11\epsilon_0 E_0 \vec{i}_x + 8\epsilon_0 E_0 \vec{i}_y$, $\vec{D}$ not $\parallel \vec{E}$

HOMEWORK PROBLEM (Due 10/25/74)

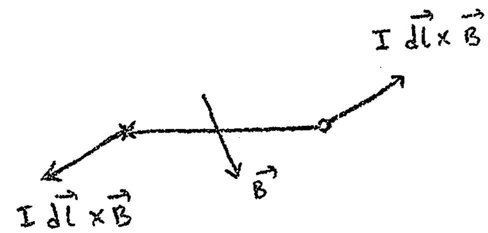
In the above example, if $\vec{E} = E_x \vec{i}_x + E_y \vec{i}_y$, find the value(s) of E_y/E_x for which $\vec{D}$ is parallel to $\vec{E}$.

MAGNETIC MATERIALS

Magnetic materials are characterized by magnetization. Magnetization is the phenomenon by means of which the orbital and spin motions of electrons are influenced by an applied magnetic field, to create a net magnetic dipole moment.



Magnetic dipole



Torque acting on a magnetic dipole gives rise to paramagnetism

Diamagnetism in which a net dipole moment is induced by change in angular velocities of electronic orbits is prevalent in all materials.

The strength of a magnetic dipole is defined by the magnetic dipole moment $\vec{m}$ given by

$$\vec{m} = I A \vec{i}_n \qquad \text{amperes - meter}^2$$

where A is the area of the current loop and $\vec{i}_n$ is defined in the right hand sense.

On a macroscopic scale, we define

$$\vec{M} = \frac{1}{\Delta v} \sum_{j=1}^{N\Delta v} \vec{m}_j = N \vec{m} \qquad \frac{\text{Ampere - meter}^2}{\text{meter}^3} \text{ or } \frac{\text{Ampere}}{\text{meter}}$$

where N is the number of molecules per unit volume of the material and $\vec{m}$ is the average dipole moment per molecule.

$\vec{M}$ is known as the "<u>magnetization vector</u>" or the magnetic dipole moment per unit volume.

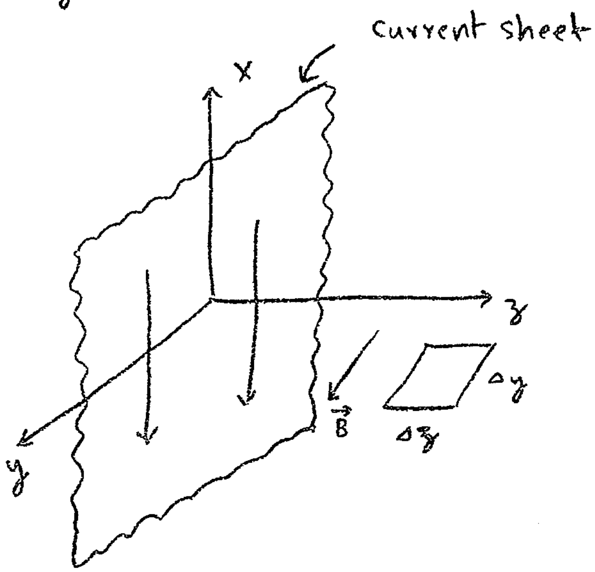
For many magnetic materials, $\vec{M}$ is related to $\vec{B}$ in the manner

$$\vec{M} = \frac{\chi_m}{1+\chi_m}\frac{\vec{B}}{\mu_0}$$

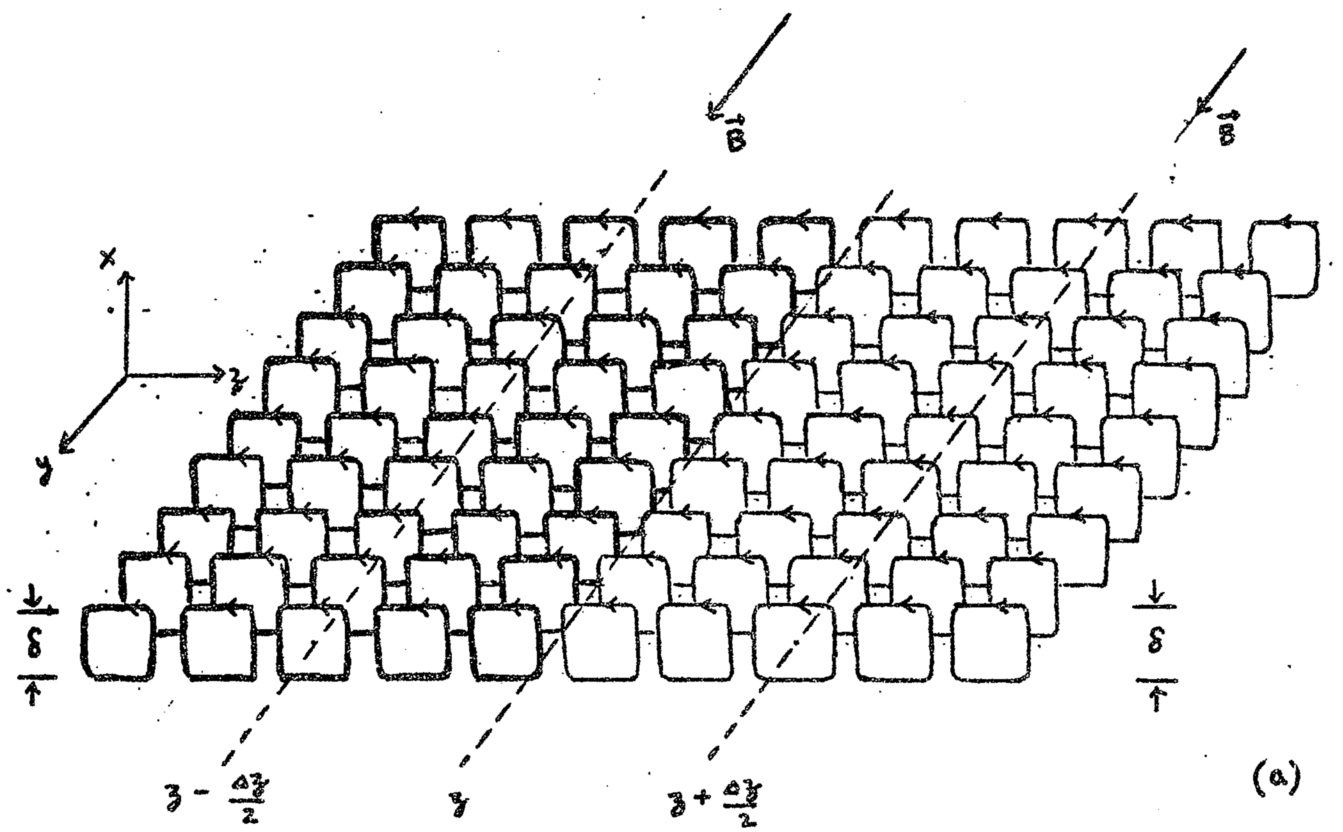
where χ_m, a dimensionless parameter known as the "magnetic susceptibility" is a measure of the ability of the material to become magnetized and differs from one magnetic material to another.

Let us consider the infinite plane current sheet with the space on either side of it now possessing magnetic material properties in addition to any dielectric properties. Then the magnetic field induces magnetization in the material as given by

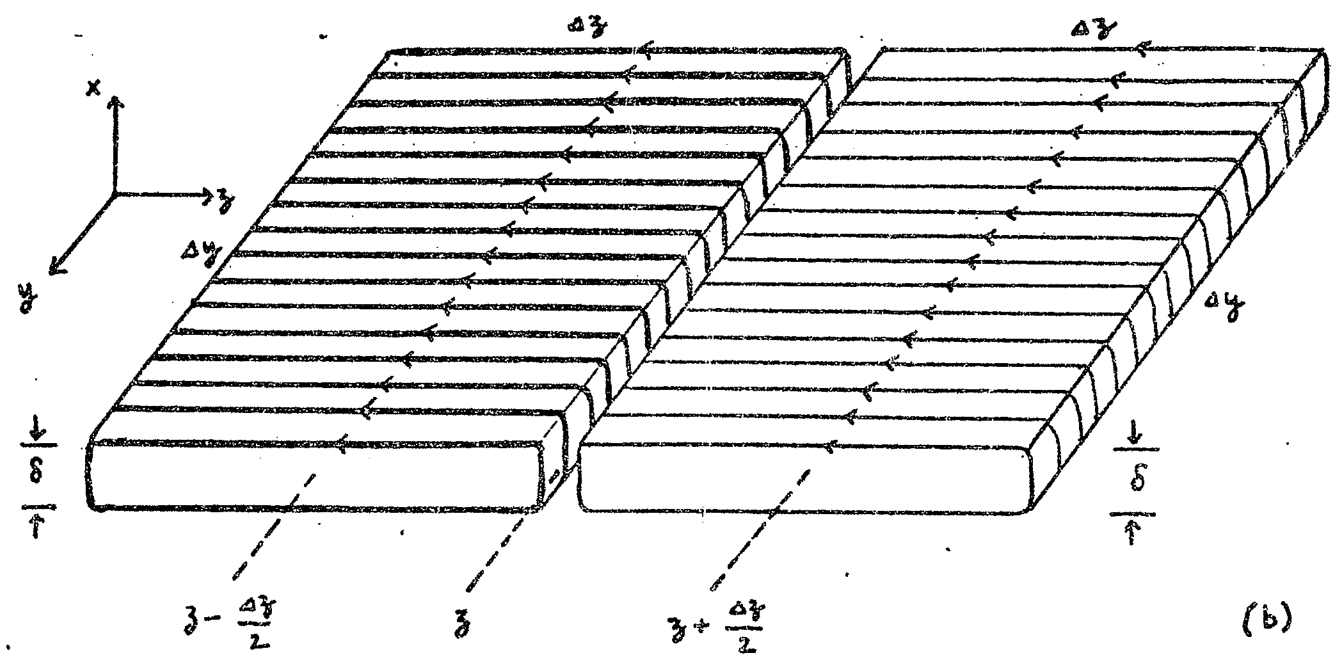
$$\vec{M} = M_y \vec{i}_y = \frac{\chi_m}{1+\chi_m}\frac{B_y}{\mu_0}\vec{i}_y$$



Note that B_y is a function of z and t.
If we now consider an infinitesimal area $\Delta y\,\Delta z$ as shown in the figure, the magnetic dipole moments induced by the magnetic field can be pictured as shown on the following page.


$\vec{B}$
$\vec{B}$
x
z
y
δ
δ
$z - \frac{\Delta z}{2}$
z
$z + \frac{\Delta z}{2}$

(a)


x
z
y
Δz
Δz
Δy
Δy
δ
δ
$z - \frac{\Delta z}{2}$
z
$z + \frac{\Delta z}{2}$

(b)

The situation can be considered as two rectangular surface current loops as shown in figure (b). Since B_y is a function of z, the surface currents in the two loops are not equal and hence there is a net current flowing in the x direction. This current is known as the "<u>magnetization current</u>". To find the magnetization current crossing the surface, we note that

the total dipole moment associated with the dipoles in the left area

$$= [M_y]_{z-\frac{\Delta z}{2}} (\delta \, \Delta y \, \Delta z) = [M_y]_{z-\frac{\Delta z}{2}} \Delta y \; (\delta \, \Delta z)$$

= surface current of the left loop $\times$ cross sectional area

$\therefore$ Surface current of the left loop $= [M_y]_{z-\frac{\Delta z}{2}} \Delta y$

Similarly,

the total dipole moment associated with the dipoles in the right area

$$= [M_y]_{z+\frac{\Delta z}{2}} (\delta \, \Delta y \Delta z) = [M_y]_{z+\frac{\Delta z}{2}} \Delta y \; (\delta \, \Delta z)$$

= surface current of the right loop $\times$ crosssectional area

$\therefore$ Surface current of the right loop $= [M_y]_{z+\frac{\Delta z}{2}} \Delta y$

Thus, the magnetization crossing the surface in the x direction is

$$I_{mx} = [M_y]_{z-\frac{\Delta z}{2}} \Delta y - [M_y]_{z+\frac{\Delta z}{2}} \Delta y$$

$$= -\left\{ [M_y]_{z+\frac{\Delta z}{2}} - [M_y]_{z-\frac{\Delta z}{2}} \right\} \Delta y$$

or the magnetization current density is given by

$$J_{mx} = \frac{I_{mx}}{\Delta y \Delta z} = -\frac{\left[(M_y)_{z+\frac{\Delta z}{2}} - (M_y)_{z-\frac{\Delta z}{2}}\right]\Delta y}{\Delta y \, \Delta z}$$

$$= -\frac{\partial M_y}{\partial z}$$

or

$$\boxed{\vec{J}_m = \vec{\nabla} \times \vec{M}}$$

$$\vec{\nabla} \times \vec{M} = \begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & M_y & 0 \end{vmatrix} = -\frac{\partial M_y}{\partial z}\vec{i}_x$$

In considering electromagnetic wave propagation in magnetic material medium, the magnetization current density must be taken into account on the right side of Ampere's circuital law.

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \vec{J} + \vec{J}_m + \frac{\partial \vec{D}}{\partial t}$$

$$= \vec{J} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{D}}{\partial t}$$

($\vec{J}$ represents any other current density which may be present in addition to the magnetization and polarization current densities)

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

We now redefine $\vec{H}$ to read as

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$= \frac{\vec{B}}{\mu_0} - \frac{\chi_m}{1+\chi_m}\frac{\vec{B}}{\mu_0}$$

$$= \vec{B}/(1+\chi_m)\mu_0$$

$$= \vec{B}/\mu_r\mu_0$$ where $\mu_r = 1+\chi_m$ is the relative permeability

$$\boxed{\vec{H} = \vec{B}/\mu}$$ where $\mu = \mu_0\mu_r$ is the permeability

so that by replacing μ_0 by μ, we can avoid taking into account explicitly the magnetization current density! The quantity μ takes into account the effects of magnetization and we can forget about them.

ADDITIONAL PROBLEMS

1. In a region containing no charges and currents, the magnetic field is given by

$$\vec{B} = B_0 \sin \beta z \sin \omega t \ \vec{i}_x$$

where B_0, β and ω are constants. Using one of Maxwell's curl equations at a time, find two expressions for the associated electric field $\vec{E}$ and then find the relationship between β, ω, μ_0 and ϵ_0.

2. For the vector

$$\vec{A} = 2xy \ \vec{i}_x + x^2 \vec{i}_y + \vec{i}_z ,$$

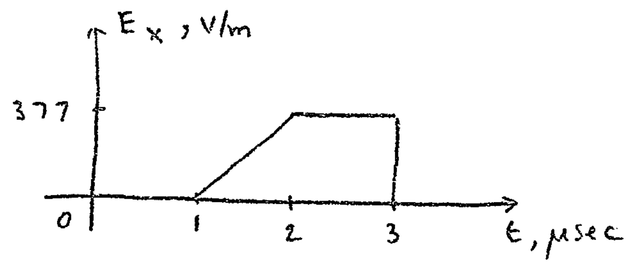
show that the line integral between any two points P and Q is independent of the path. Then evaluate $\int_{(0,0,0)}^{(1,1,1)} \vec{A} \cdot d\vec{l}$.

3. For the infinite plane current sheet situated in the plane $z = 0$, the electric field E_x observed in the plane $z = 300$ m is found to be as shown below. Find and sketch

(a) E_x vs. t for $z = 600$ m

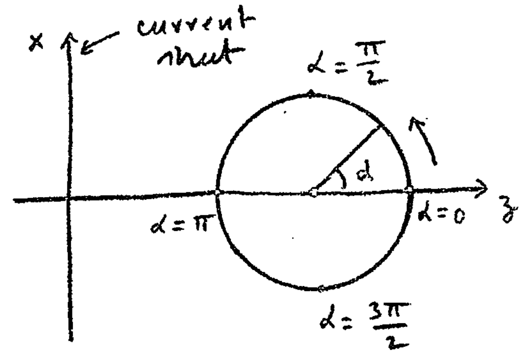
(b) E_x vs. z for $t = 1$ μsec

(c) H_y vs. z for $t = 2$ μsec.

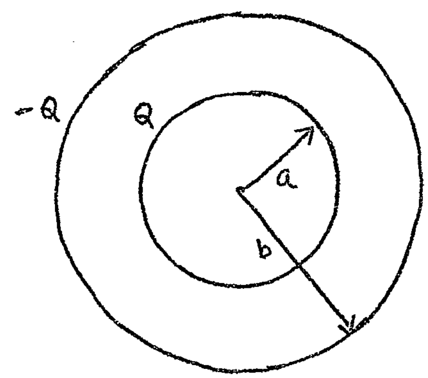


4. Consider an observer moving uniformly on the circumference of a circle with an angular ~~frequency~~ velocity ω_0 rad/sec, in the field of a uniform plane wave of frequency ω propagating away from an infinite plane current sheet.

Find and sketch the doppler shift observed by the moving observer as a function of the angle α.



5. Find the energy stored in an arrangement of two concentric spherical charges Q and −Q coulombs distributed uniformly on spherical surfaces of radii a and b.



HOUR EXAM NO. 2 : WEDNESDAY OCTOBER 30.

UNIFORM PLANE WAVES IN A MATERIAL MEDIUM:

For a material medium characterized by conductivity σ, permittivity ϵ and permeability μ,

$$\boxed{\begin{aligned} \vec{J}_c &= \sigma \vec{E} \\ \vec{D} &= \epsilon \vec{E} \\ \vec{H} &= \vec{B}/\mu \end{aligned}}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

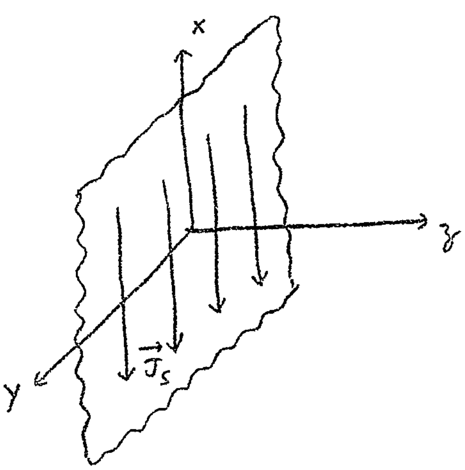
For the simple case of

$$\vec{E} = E_x(z,t)\,\vec{i}_x$$

$$\vec{H} = H_y(z,t)\,\vec{i}_y$$

as for the electromagnetic field due to the infinite plane current sheet in the $z=0$ plane and carrying uniformly distributed current in the negative x direction, we have

$$\boxed{\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial H_y}{\partial z} &= -\sigma E_x - \epsilon \frac{\partial E_x}{\partial t} \end{aligned}}$$



For sinusoidally time varying fields, we make use of phasor notation to write

$$E_x(z;t) = \mathrm{Re}\left[\bar{E}_x(z)\cdot e^{j\omega t}\right]$$

$$H_y(z,t) = \mathrm{Re}\left[\bar{H}_y(z)\cdot e^{j\omega t}\right]$$

Replacing E_x and H_y in the differential equations by their phasors and $\frac{\partial}{\partial t}$ by $j\omega$, we obtain the corresponding differential equations for the phasors $\bar{E}_x$ and $\bar{H}_y$ as

$$\frac{\partial \bar{E}_x}{\partial z} = -j\omega\mu\,\bar{H}_y \qquad \text{--- (1)}$$

$$\frac{\partial \bar{H}_y}{\partial z} = -\sigma\bar{E}_x - j\omega\epsilon\bar{E}_x = -(\sigma + j\omega\epsilon)\,\bar{E}_x \qquad \text{--- (2)}$$

Differentiating (1) with respect to z and using (2), we obtain

$$\frac{\partial^2 \bar{E}_x}{\partial z^2} = -j\omega\mu\frac{\partial \bar{H}_y}{\partial z} = -j\omega\mu\left[-(\sigma + j\omega\epsilon)\,\bar{E}_x\right]$$

$$= j\omega\mu(\sigma + j\omega\epsilon)\,\bar{E}_x$$

Defining

$$\boxed{\bar{\gamma} = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}\,,}$$

we have

$$\boxed{\frac{\partial^2 \bar{E}_x}{\partial z^2} = \bar{\gamma}^2\,\bar{E}_x} \qquad \text{--- (3)}$$

Eq. (3) is the wave equation for $\bar{E}_x$ in the material medium.

Its solution is given by

$$\bar{E}_x(z) = \bar{A}\, e^{-\bar{\gamma} z} + \bar{B}\, e^{\bar{\gamma} z} \qquad \text{— (4)}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants.

Noting that $\bar{\gamma}$ is a complex number and hence can be written as

$$\bar{\gamma} = \alpha + j\beta, \qquad \text{— (5)}$$

we have

$$\bar{E}_x(z) = \bar{A}\, e^{-\alpha z} e^{-j\beta z} + \bar{B}\, e^{\alpha z} e^{j\beta z}$$

$$= A\, e^{j\theta} e^{-\alpha z} e^{-j\beta z} + B\, e^{j\phi} e^{\alpha z} e^{j\beta z}$$

$$E_x(z,t) = \mathcal{R}e\left[\bar{E}_x(z)\, e^{j\omega t}\right]$$

$$= \mathcal{R}e\left[A\, e^{j\theta} e^{-\alpha z} e^{-j\beta z} e^{j\omega t} + B\, e^{j\phi} e^{\alpha z} e^{j\beta z} e^{j\omega t}\right]$$

$$= A\, e^{-\alpha z}\cos(\omega t - \beta z + \theta) + B\, e^{\alpha z}\cos(\omega t + \beta z + \phi) \qquad \text{— (6)}$$

We recognize that the first term represents a (+) wave, i.e., a wave traveling in the positive z direction and the second term represents a (−) wave, i.e, a wave traveling in the negative z direction. Also, since α is a positive number in view of the fact that the phase angle of $\bar{\gamma}\left(= \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}\right)$ is less than $90°$, the (+) and (−) waves get attenuated as they propagate in their respective directions.

Thus the solution for the electric field on either side of the sheet is

$$E_x(z,t) = \begin{cases} A\, e^{-\alpha z} \cos(\omega t - \beta z + \theta) & \text{for } z>0 \\ B\, e^{\alpha z} \cos(\omega t + \beta z + \phi) & \text{for } z<0 \end{cases} \quad (7)$$

The quantity α is known as the attenuation constant. The attenuation per unit length is $e^{(\alpha)(1)}$ or e^{α}. In terms of decibels, this is equal to $20 \log_{10} e^{\alpha}$ or $8.68\,\alpha$ db. The units of α are Nepers/m.

The quantity β is of course the phase constant, i.e., the rate of change of phase with z for fixed t.

The quantity $\bar{\gamma}$ is known as the propagation constant since its real and imaginary parts, i.e., α and β together determine propagation characteristics, i.e., attenuation and phase shift of the wave.

To find expressions for α and β in terms of ω, σ, μ and ϵ, we have from (5) and (6)

$$(\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\epsilon)$$

$$\alpha^2 - \beta^2 + 2j\alpha\beta = -\omega^2\mu\epsilon + j\,\omega\mu\sigma \quad (8)$$

Equating real and imaginary parts on either side,

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \quad (9)$$

$$2\alpha\beta = \omega\mu\sigma \quad (10)$$

squaring (9) and (10) and adding and then taking the square root, we obtain

$$\alpha^2 + \beta^2 = \sqrt{\omega^4 \mu^2 \epsilon^2 + \omega^2 \mu^2 \sigma^2} = \omega^2 \mu \epsilon \sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} \qquad (11)$$

From (9) and (10), we then have

$$\alpha = \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} - 1 \right]^{1/2} \qquad (12)$$

$$\beta = \frac{\omega \sqrt{\mu \epsilon}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1 \right]^{1/2} \qquad (13)$$

We note that both α and β are dependent on the factor $\sigma/\omega\epsilon$. This factor, known as the <u>loss tangent</u>, is the ratio of the magnitude of the conduction current density σE_x to the magnitude of the displacement current density, $\omega \epsilon E_x$.

We also note that

$$v_p = \frac{\omega}{\beta} = \sqrt{\frac{2}{\mu \epsilon}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1 \right]^{-1/2} \qquad (14)$$

is dependent on frequency. This feature gives rise to "dispersion".

The wavelength in the medium is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f \sqrt{\mu \epsilon}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon}\right)^2} + 1 \right]^{-1/2} \qquad (15)$$

we now find the solution for the magnetic field by substituting (4) into (1). Thus

$$-j\omega\mu \bar{H}_y = \frac{\partial \bar{E}_x}{\partial z} = -\bar{\gamma}\bar{A}e^{-\bar{\gamma}z} + \bar{\gamma}\bar{B}e^{\bar{\gamma}z}$$

$$\bar{H}_y = \frac{\bar{\gamma}}{j\omega\mu}\left[\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}\right]$$

$$= \sqrt{\frac{\sigma + j\omega\epsilon}{j\omega\mu}}\left[\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}\right]$$

$$= \frac{1}{\bar{\eta}}\left[\bar{A}e^{-\bar{\gamma}z} - \bar{B}e^{\bar{\gamma}z}\right] \qquad (16)$$

where

$$\boxed{\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}}$$ is the intrinsic impedance of the medium. (17)

Writing

$$\bar{\eta} = |\bar{\eta}|\, e^{j\delta},$$

we obtain the solution for $H_y(z,t)$ as

$$H_y(z,t) = \mathrm{Re}\left[\bar{H}_y(z)\, e^{j\omega t}\right]$$

$$= \mathrm{Re}\left[\frac{1}{|\bar{\eta}|e^{j\delta}} A e^{j\theta} e^{-\alpha z} e^{-j\beta z} e^{j\omega t} - \frac{1}{|\bar{\eta}|e^{j\delta}} B e^{j\phi} e^{\alpha z} e^{j\beta z} e^{j\omega t}\right]$$

$$= \frac{A}{|\bar{\eta}|} e^{-\alpha z}\cos(\omega t - \beta z + \theta - \delta) - \frac{B}{|\bar{\eta}|} e^{\alpha z}\cos(\omega t + \beta z + \phi - \delta) \qquad (18)$$

By comparing (18) with (6), we note that E_x and H_y are not in phase.

Recalling that the solution for H_y adjacent to the current sheet is given by

$$H_y = \begin{cases} \frac{J_{S0}}{2} \cos \omega t & \text{for } z = 0+ \\ -\frac{J_{S0}}{2} \cos \omega t & \text{for } z = 0- \end{cases}$$

and comparing with (18), we obtain

$$A = \frac{|\bar{\eta}| J_{S0}}{2}, \quad \theta = \delta$$

$$B = \frac{|\bar{\eta}| J_{S0}}{2}, \quad \phi = \delta$$

Thus the electromagnetic field due to the infinite plane current sheet

$$\vec{J}_S = -J_{S0} \cos \omega t \, \vec{i}_x \quad \text{on } z = 0$$

with a material medium characterized by σ, ϵ and μ situated on either side of it is given by

$$\boxed{\begin{aligned} E_x(z,t) &= \begin{cases} \frac{|\bar{\eta}| J_{S0}}{2} e^{-\alpha z} \cos(\omega t - \beta z + \delta) & \text{for } z > 0 \\ \frac{|\bar{\eta}| J_{S0}}{2} e^{\alpha z} \cos(\omega t + \beta z + \delta) & \text{for } z < 0 \end{cases} \\ H_y(z,t) &= \begin{cases} \frac{J_{S0}}{2} e^{-\alpha z} \cos(\omega t - \beta z + \delta) & \text{for } z > 0 \\ -\frac{J_{S0}}{2} e^{\alpha z} \cos(\omega t + \beta z + \delta) & \text{for } z < 0 \end{cases} \end{aligned}}$$

HOMEWORK PROBLEM (Due 11/1/74)

Obtain the expression for the attenuation constant per wavelength in the material medium characterized by σ, μ and ϵ. Using a logarithmic scale for $\frac{\sigma}{\omega\epsilon}$, plot the attenuation per wavelength versus $\frac{\sigma}{\omega\epsilon}$.

PROPAGATION CHARACTERISTICS IN MATERIAL MEDIA :

For a material medium characterized by conductivity σ, permittivity ϵ, and permeability μ, we found that

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right]^{1/2}$$

$$\beta = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]^{1/2}$$

$$v_p = \frac{\sqrt{2}}{\sqrt{\mu\epsilon}} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]^{-1/2}$$

$$\lambda = \frac{\sqrt{2}}{f\sqrt{\mu\epsilon}} \left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]^{-1/2}$$

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

Case 1 : Perfect dielectric, $\sigma = 0$

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= \omega\sqrt{\mu\epsilon} \\ v_p &= 1/\sqrt{\mu\epsilon} \\ \lambda &= 1/f\sqrt{\mu\epsilon} \\ \bar{\eta} &= \sqrt{\mu/\epsilon} \end{aligned}\right\}$$

No attenuation

Same as in the case of free space except that ϵ_0 and μ_0 are replaced by ϵ and μ, respectively.

Case 2: Good dielectric, $\sigma/\omega\epsilon \ll 1$

Good dielectrics are characterized by conduction current small in comparison with displacement current, i.e., $|\sigma \bar{E}_x| \ll |j\omega\epsilon \bar{E}_x|$, or, $\sigma \ll \omega\epsilon$, or $\sigma/\omega\epsilon \ll 1$. Then

$$\alpha = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \left[1 + \frac{\sigma^2}{2\omega^2\epsilon^2} - \frac{\sigma^4}{8\omega^4\epsilon^4} + \cdots \quad -1 \right]^{1/2}$$

$$\approx \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \frac{\sigma}{\sqrt{2}\,\omega\epsilon} \left[1 - \frac{\sigma^2}{4\omega^2\epsilon^2} \right]^{1/2}$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2} \right)$$

$$\beta \approx \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \left[2 + \frac{\sigma^2}{2\omega^2\epsilon^2} \right]^{1/2}$$

$$= \omega\sqrt{\mu\epsilon} \left(1 + \frac{\sigma^2}{8\omega^2\epsilon^2} \right)$$

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\epsilon}} \left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2} \right)$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{1}{f\sqrt{\mu\epsilon}} \left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2} \right)$$

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{j\omega\epsilon}} \left[1 - j\frac{\sigma}{\omega\epsilon} \right]^{-1/2}$$

$$= \sqrt{\frac{\mu}{\epsilon}} \left[1 + j\frac{\sigma}{2\omega\epsilon} - \frac{3}{8}\frac{\sigma^2}{\omega^2\epsilon^2} - \cdots \right]$$

$$\approx \sqrt{\frac{\mu}{\epsilon}} \left[\left(1 - \frac{3}{8}\frac{\sigma^2}{\omega^2\epsilon^2} \right) + j\frac{\sigma}{2\omega\epsilon} \right]$$

Case 3: Good Conductor, $\sigma/\omega\epsilon \gg 1$

Good conductors are characterized by conduction current large in comparison with displacement current, i.e., $|\sigma \bar{E}_x| \gg |j\omega\epsilon \bar{E}_x|$, or, $\sigma \gg \omega\epsilon$, or $\sigma/\omega\epsilon \gg 1$. Then

$$\alpha \approx \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}} \sqrt{\frac{\sigma}{\omega\epsilon}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma}$$

$$\beta \approx \frac{\omega\sqrt{\mu\epsilon}}{2} \sqrt{\frac{\sigma}{\omega\epsilon}} = \sqrt{\pi f \mu \sigma}$$

$$v_p \approx \frac{\sqrt{2}}{\sqrt{\mu\epsilon}} \sqrt{\frac{\omega\epsilon}{\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{4\pi f}{\mu\sigma}}$$

$$\lambda \approx \frac{\sqrt{2}}{f\sqrt{\mu\epsilon}} \sqrt{\frac{\omega\epsilon}{\sigma}} = \sqrt{\frac{4\pi}{f\mu\sigma}}$$

$$\bar{\eta} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}.$$ Note that $\sqrt{j} = \frac{1+j}{\sqrt{2}}$

Let us consider the example of copper. The constants of copper are: $\sigma = 5.8 \times 10^7$ mhos/m, $\epsilon = \epsilon_0$, and $\mu = \mu_0$.

Hence the frequency at which $\sigma/\omega\epsilon$ is equal to 1 is given by

$$f = \frac{\sigma}{2\pi\epsilon} = \frac{5.8 \times 10^7}{2\pi \times 10^{-9}/36\pi} = 18 \times 5.8 \times 10^{16} = 1.04 \times 10^{18} \text{ Hz}.$$

Thus at frequencies of even several GHz, copper behaves like an excellent conductor.

Attenuation undergone in one wavelength $= e^{-\alpha\lambda}$

$$= e^{-\sqrt{\pi f \mu \sigma}\cdot\sqrt{4\pi/f\mu\sigma}} = e^{-2\pi}$$

In terms of db, this is $20\log_{10} e^{2\pi}$ or 54.5 db.

In fact, the field is attenuated by a factor e^{-1} or 0.368 in a distance of $1/\alpha$. This distance is known as the skin depth, δ. Thus

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}.$$

For copper, $\delta = \dfrac{1}{\sqrt{\pi f \times 4\pi\times10^{-7}\times 5.8\times10^{7}}} = \dfrac{0.066}{\sqrt{f}}$ m

For $f = 1$ MHz, $\delta_{copper} = \dfrac{0.066}{10^3}$ m $= 0.066$ mm.

Thus even at a low frequency of 1 MHz, the fields in copper are attenuated by the factor e^{-1} in a distance of 0.066 mm. This phenomenon is known as the "skin effect" and also explains "shielding" by conductors.

Case 4: Perfect Conductors, $\sigma = \infty$

Perfect conductors are idealizations of good conductors. Since $\sigma = \infty$, there cannot be any fields inside a perfect conductor.

EXAMPLE: For uniform plane wave propagation in sea water ($\sigma = 4$ mhos/m, $\epsilon = 80\epsilon_0$, $\mu = \mu_0$), find $\alpha, \beta, v_p, \lambda$ and $\bar{\eta}$ for two frequencies: (a) 10,000 MHz and (b) 25 kHz.

The frequency at which $\sigma/\omega\epsilon = 1$ is given by

$$\frac{\sigma}{2\pi\epsilon} = \frac{4}{2\pi \times 80 \times 10^{-9}/36\pi} = \frac{72}{80 \times 10^{-9}} = 0.9 \times 10^9 \, Hz = 900 \, MHz.$$

Hence for 10,000 MHz, $\sigma/\omega\epsilon \ll 1$, sea water is a good dielectric and for 25 kHz, $\sigma/\omega\epsilon \gg 1$, sea water is a good conductor.

Thus, for 10,000 MHz,

$$\alpha \approx \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2}\right) \approx \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2} \times 4 \times \sqrt{\frac{\mu_0}{80\epsilon_0}} = 2 \times \frac{377}{\sqrt{80}}$$

$$= 84.3 \text{ nepers/m}$$

$$\beta \approx \omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma^2}{8\omega^2\epsilon^2}\right) \approx \omega\sqrt{\mu\epsilon} = \frac{2\pi \times 10^{10} \times \sqrt{80}}{3 \times 10^8}$$

$$= 1873 \text{ rad/m}$$

$$v_p \approx \frac{1}{\sqrt{\mu\epsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2}\right) \approx \frac{1}{\sqrt{\mu\epsilon}} = \frac{3 \times 10^8}{\sqrt{80}} = 0.3354 \times 10^8 \text{ m/sec}$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{1873} = 3.353 \times 10^{-3} \text{ m} = 3.353 \text{ mm}$$

as compared to 30 mm in free space

$$\bar{\eta} \approx \sqrt{\frac{\mu}{\epsilon}}\left[\left(1 - \frac{3}{8}\frac{\sigma^2}{\omega^2\epsilon^2}\right) + j\frac{\sigma}{2\omega\epsilon}\right] \approx \sqrt{\frac{\mu}{\epsilon}} = \frac{377}{\sqrt{80}} = 42.15 \text{ ohms.}$$

For 25 kHz,

$$\alpha \approx \sqrt{\pi f \mu \sigma} = \sqrt{\pi \times 25 \times 10^3 \times 4\pi \times 10^{-7} \times 4} = 0.2\pi \text{ nepers/m}$$

$$\beta \approx \sqrt{\pi f \mu \sigma} = \alpha = 0.2\pi \text{ rad/m}$$

$$v_p \approx \sqrt{\frac{4\pi f}{\mu \sigma}} = \sqrt{\frac{4\pi \times 25 \times 10^3}{4\pi \times 10^{-7} \times 4}} = 2.5 \times 10^5 \text{ m/sec}$$

$$\lambda = \frac{2\pi}{\beta} \approx \frac{2\pi}{0.2\pi} = 10 \text{ m} \quad \left(\text{as compared to } 12 \text{ km in free space}\right)$$

$$\bar{\eta} \approx (1+j)\sqrt{\frac{\pi f \mu}{\sigma}} = (1+j)\sqrt{\frac{\pi \times 25 \times 10^3 \times 4\pi \times 10^{-7}}{4}}$$

$$= 0.05\pi\,(1+j) \text{ ohms.}$$

It can be seen that in view of the low attenuation and small wavelength, low frequencies are more suitable for communication under water.

HOMEWORK PROBLEM (Due 11/6/74)

For uniform plane wave propagation in fresh lake water ($\sigma = 10^{-3}$ mho/m, $\epsilon = 80\epsilon_0$, $\mu = \mu_0$), find α, β, v_p, λ and $\bar{\eta}$ for two frequencies: (a) 100 MHz and (b) 10 kHz.

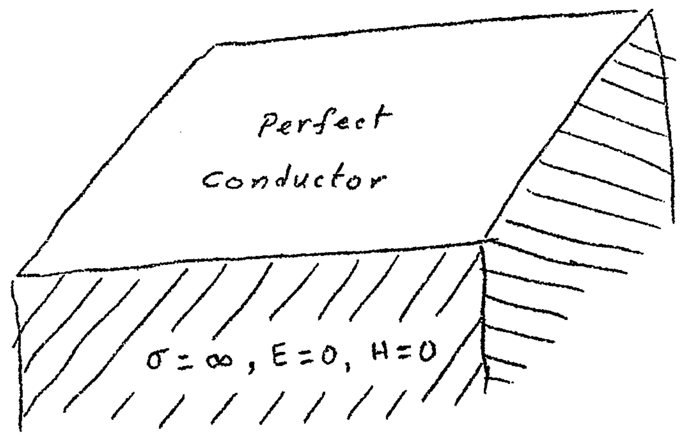
BOUNDARY CONDITIONS ON A PERFECT CONDUCTOR SURFACE:

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int \vec{D}\cdot d\vec{s}$$

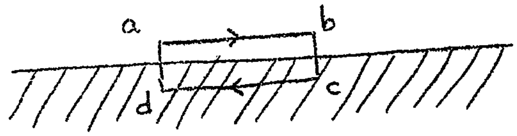
$$\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv$$

$$\oint \vec{B}\cdot d\vec{s} = 0$$



(1) $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$

Choosing a rectangular path abcda such that ab and cd are very close to and on either side of the boundary and applying Faraday's law in integral form, we obtain



$$\oint_{abcda} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{abcd} \vec{B}\cdot d\vec{s}$$

$$\int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l} + \int_c^d \vec{E}\cdot d\vec{l} + \int_d^a \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_{abcd} \vec{B}\cdot d\vec{s}$$

But $\int_c^d \vec{E}\cdot d\vec{l} = 0$ since $\vec{E} = 0$ inside the perfect conductor.

If we now let ad and bc $\to 0$, i.e., let ab and cd almost touch each other but remaining on either side of the boundary,

$$\int_b^c \vec{E} \cdot d\vec{l} \to 0 \quad , \quad \int_d^a \vec{E} \cdot d\vec{l} \to 0 \quad \text{and} \quad \int_{abcd} \vec{B} \cdot d\vec{s} \to 0 .$$

Thus we obtain

$$\int_a^b \vec{E} \cdot d\vec{l} = 0$$

$$E_{ab}\,(ab) = 0$$

or $E_{ab} = 0$

Since we can choose the rectangle abcda with any orientation, it follows that $E_{ab} = 0$ for any orientation of ab.

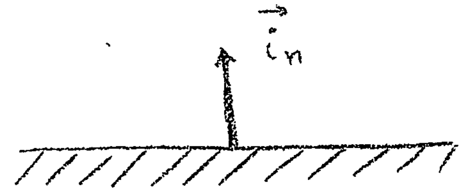
B.C. I

Hence, the tangential component of $\vec{E}$ on a perfect conductor surface is equal to zero.

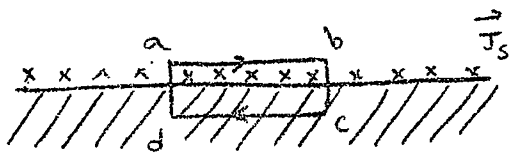
In vector form, we have

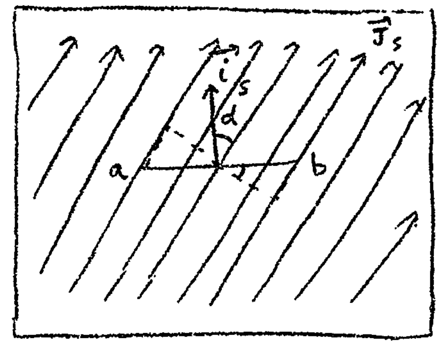
$$\boxed{\vec{i}_n \times \vec{E} = 0}$$ on the conductor surface

where $\vec{i}_n$ is the unit normal vector to the conductor surface pointing out of the conductor.



(2) $$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int \vec{D}\cdot d\vec{s}$$





Applying Ampere's circuital law to the rectangular path abcd, we have

$$\int_a^b \vec{H}\cdot d\vec{l} + \int_b^c \vec{H}\cdot d\vec{l} + \int_c^d \vec{H}\cdot d\vec{l} + \int_d^a \vec{H}\cdot d\vec{l} = \int_{abcd} \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_{abcd} \vec{D}\cdot d\vec{s}$$

Again $\int_c^d \vec{H}\cdot d\vec{l} = 0$ since $\vec{H} = 0$ inside the perfect conductor.

If we now let ab and cd $\to 0$ as before,

$$\int_b^c \vec{H}\cdot d\vec{l} \to 0, \quad \int_d^a \vec{H}\cdot d\vec{l} \to 0, \quad \int_{abcd} \vec{D}\cdot d\vec{s} \to 0, \quad \text{but}$$

$$\int_{abcd} \vec{J}\cdot d\vec{s} \longrightarrow J_s\,(ab\cos\alpha)$$

Thus we obtain

$$H_{ab}\,(ab) = J_s\,(ab\cos\alpha)$$

$$H_{ab} = J_s\cos\alpha$$

The maximum value of H_{ab}, i.e., the $H_{tangential}$ is obtained when ab is oriented perpendicular to $\vec{J}_s$ and then

$$H_t = J_s$$

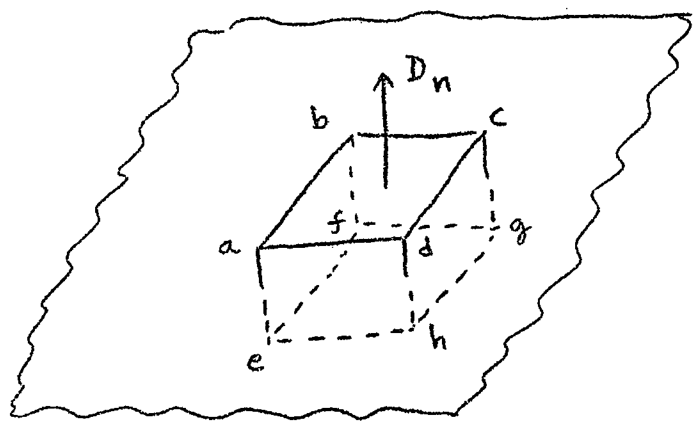
B.C. II

> Hence, the tangential component of $\vec{H}$ on a perfect conductor surface is perpendicular to the direction of the surface current density (in the right hand sense) and is equal in magnitude to the surface current density.

In vector form, we have

$\boxed{\vec{i}_n \times \vec{H} = \vec{J}_s}$ on the conductor surface.

③ $\oint_S \vec{D} \cdot d\vec{s} = \int_V \rho \, dv$



Considering a rectangular box abcdefgh such that the surfaces abcd and efgh are very close to and on either side of the boundary and applying Gauss' law for the electric field, we obtain

$$\int_{abcd} \vec{D} \cdot d\vec{s} + \int_{\substack{\text{side}\\\text{surfaces}}} \vec{D} \cdot d\vec{s} + \int_{efgh} \vec{D} \cdot d\vec{s} = \int_{\substack{\text{volume}\\\text{of the box}}} \rho \, dv$$

But $\int_{efgh} \vec{D} \cdot d\vec{s} = 0$ since $\vec{D} = 0$ inside the perfect conductor.

If we now let the side surfaces $\rightarrow 0$, i.e., let abcd and efgh almost touch each other but remaining on either side of the boundary,

$$\int_{\substack{\text{side}\\ \text{surfaces}}} \vec{D}\cdot d\vec{s} \rightarrow 0 , \quad \int_{\substack{\text{Volume}\\ \text{of the box}}} \rho\, dv \longrightarrow \rho_s\,(abcd)$$

where ρ_s is the surface charge density. Thus we obtain

$$[D_n]_{abcd}\,(abcd) = \rho_s\,(abcd)$$

$$D_n = \rho_s$$

B.C. III

Hence, the normal component of $\vec{D}$ on a perfect conductor surface is equal to the surface charge density.

In vector form, we have

$$\boxed{\vec{i}_n \cdot \vec{D} = \rho_s}$$

④ $\oint_S \vec{B}\cdot d\vec{s} = 0$

Applying this to the rectangular box as in the case of ③ and letting the side surfaces $\rightarrow 0$, we have

$$\lim_{ss\to 0}\left[\int_{abcd} \vec{B}\cdot d\vec{s} + \int_{ss} \vec{B}\cdot d\vec{s} + \int_{efgh} \vec{B}\cdot d\vec{s}\right] = 0$$

or $[B_n]_{abcd}\,(abcd) = 0$

$B_n = 0.$

B.C. IV

Hence, the normal component of $\vec{B}$ on a perfect conductor surface is equal to zero.

In vector form, we have

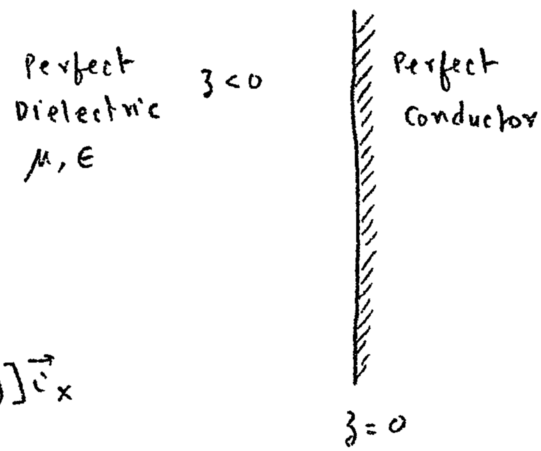
$$\boxed{\vec{i}_n \cdot \vec{B} = 0}$$

In summary, we have

$$\boxed{\begin{aligned} \vec{i}_n \times \vec{E} &= 0 \\ \vec{i}_n \times \vec{H} &= \vec{J}_s \\ \vec{i}_n \cdot \vec{D} &= \rho_s \\ \vec{i}_n \cdot \vec{B} &= 0 \end{aligned}}$$

EXAMPLE:

Consider the arrangement shown in the figure and let $\vec{E}$ in the perfect dielectric medium be given by



$$\vec{E} = [\underbrace{E_1 \cos(\omega t - \beta z)}_{(+)\text{ wave}} + \underbrace{E_2 \cos(\omega t + \beta z)}_{(-)\text{ wave}}]\vec{i}_x$$

Then since E_x is tangential to the perfect conductor surface, the boundary condition I requires that

$$[E_x]_{z=0} = 0$$

or $$[E_1 (\cos \omega t - \beta z) + E_2 \cos(\omega t + \beta z)]_{z=0} = 0$$

$$E_1 \cos \omega t + E_2 \cos \omega t = 0$$

$$E_2 = -E_1$$

Thus the electric field in the region $z<0$ must be of the form

$$\vec{E} = \left[E_1 \cos(\omega t - \beta z) - E_1 \cos(\omega t + \beta z)\right] \vec{i}_x$$

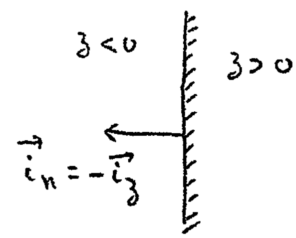
$$= 2 E_1 \sin \omega t \sin \beta z \ \vec{i}_x$$

$\vec{H}$ can then be found by using $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, or,

$$\vec{H} = \left[\frac{E_1}{\eta} \cos(\omega t - \beta z) - \frac{E_2}{\eta} \cos(\omega t + \beta z)\right] \vec{i}_y$$

$$= \left[\frac{E_1}{\eta} \cos(\omega t - \beta z) + \frac{E_1}{\eta} \cos(\omega t + \beta z)\right] \vec{i}_y$$

$$= \frac{2E_1}{\eta} \cos \omega t \cos \beta z \ \vec{i}_y$$



These solutions correspond to standing waves.

Now, from boundary condition II,

$$\left[\vec{J}_s\right]_{z=0} = \vec{i}_n \times \vec{H} = -\vec{i}_z \times \left[\vec{H}\right]_{z=0}$$

$$= -\vec{i}_z \times \frac{2E_1}{\eta} \cos \omega t \ \vec{i}_y = \frac{2E_1}{\eta} \cos \omega t \ \vec{i}_x$$

<u>HOMEWORK PROBLEM</u> (Due 11/8/74)

Two infinite, perfectly conducting sheets occupy the planes $x=0$ and $x=a$. An electric field given by

$$\vec{E} = E_0 \sin \frac{\pi x}{a} \cos \frac{\pi t}{a\sqrt{\mu_0 \epsilon_0}} \ \vec{i}_z$$

where E_0 is a constant, exists in the medium between the sheets, which is free space.

a) Show that $\vec{E}$ satisfies boundary condition I on the two sheets.

b) Obtain the magnetic field associated with the given $\vec{E}$.

c) Determine the surface current densities on the two plates.

PARALLEL PLATE TRANSMISSION LINE

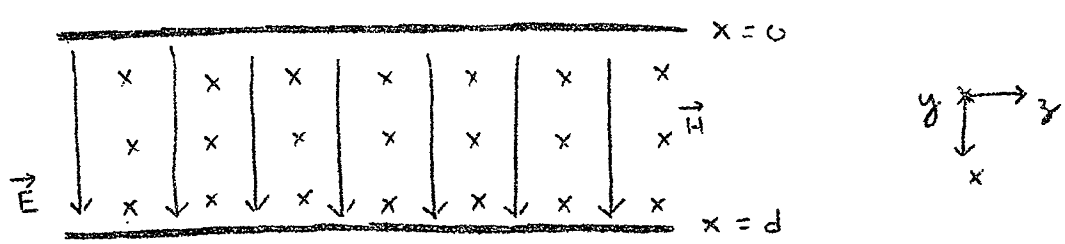
In the previous lecture, we learned that on a perfect conductor surface, the tangential component of the electric field and the normal component of the magnetic field are zero.

If we now consider the uniform plane electromagnetic wave having

$$\vec{E} = E_x(z,t)\ \vec{i}_x$$

$$\vec{H} = H_y(z,t)\ \vec{i}_y$$

and place perfectly conducting sheets in two planes $x = 0$ and $x = d$, the wave will simply be guided without any change since the two boundary condition referred to above are satisfied. We then have a simple case of transmission line, viz. the parallel plate transmission line.



According to the remaining two boundary conditions, there must be charges and currents on the conductors.

These boundary conditions are

$$\vec{i}_n \cdot \vec{D} = \rho_s$$

$$\vec{i}_n \times \vec{H} = \vec{J}_s$$

Thus the charge densities on the two plates are

$$[\rho_s]_{x=0} = \vec{i}_x \cdot \epsilon E_x \vec{i}_x = \epsilon E_x$$

$$[\rho_s]_{x=d} = (-\vec{i}_x) \cdot \epsilon E_x \vec{i}_x = -\epsilon E_x$$

where ϵ is the permittivity of the medium.

The current densities on the two plates are

$$[\vec{J}_s]_{x=0} = \vec{i}_x \times H_y \vec{i}_y = H_y \vec{i}_z$$

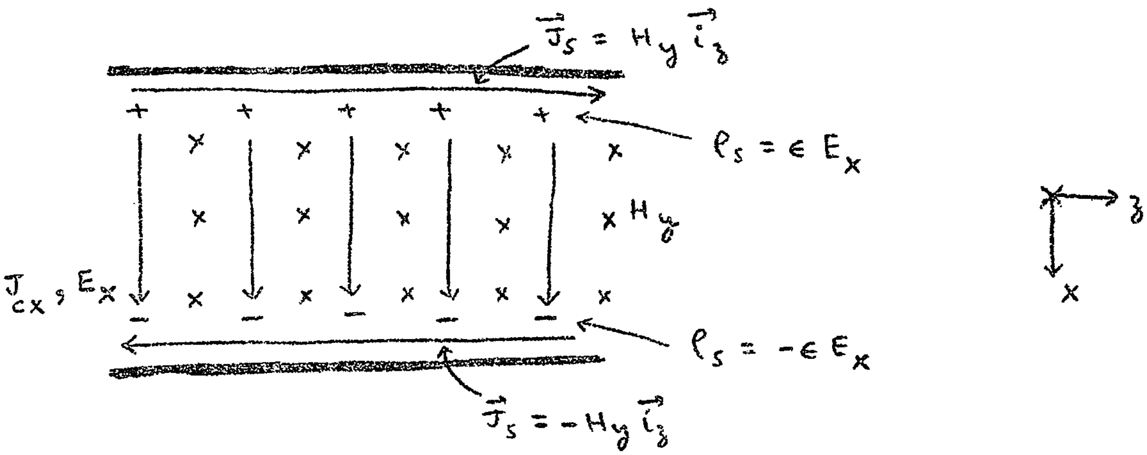
$$[\vec{J}_s]_{x=d} = (-\vec{i}_x) \times H_y \vec{i}_y = -H_y \vec{i}_z$$

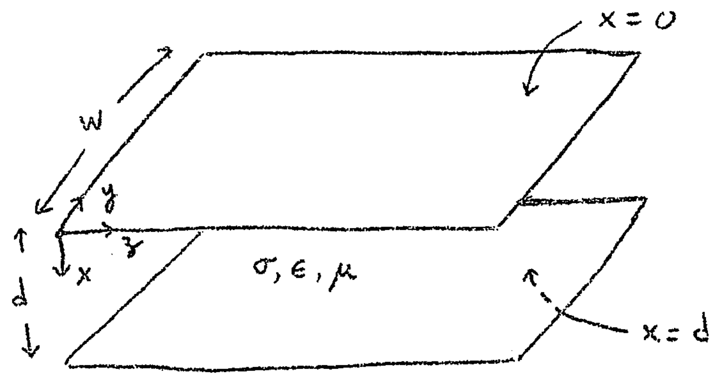
In addition there is conduction current in the medium with density given by

$$\vec{J}_c = \sigma \vec{E} = \sigma E_x \vec{i}_x$$

where σ is the conductivity of the medium.

In the above equations, it is understood that the charge and current densities are functions of z and t. Thus the wave propagation along the transmission line is supported by charges and currents varying with time and distance along the line.





Let us now consider finite size plates having width w in the y direction, as shown above, and neglect fringing of fields.

We define the voltage between the two plates for any value of z as

$$V(z,t) = \int_{x=0}^{d} E_x(z,t)\, dx = E_x(z,t)\int_{x=0}^{d} dx = d\, E_x(z,t).$$

The current flowing on either plate for any value of z is

$$I(z,t) = \int_{y=0}^{w} J_s(z,t)\, dy = \int_{y=0}^{w} H_y(z,t)\, dy = H_y(z,t)\int_{y=0}^{w} dy$$

$$= w\, H_y(z,t).$$

Thus we have

$$E_x = \frac{V}{d} \qquad \text{and} \qquad H_y = \frac{I}{w}$$

we now recall that E_x and H_y satisfy the two differential equations (simplified forms of Maxwell's curl equations)

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_y}{\partial z} = -J_{cx} - \frac{\partial D_x}{\partial t} = -\sigma E_x - \epsilon \frac{\partial E_x}{\partial t}$$

Replacing E_x and H_y by V/d and I/w respectively, we now convert these equations to be in terms of voltage and current. Thus,

$$\frac{\partial}{\partial z}\left(\frac{V}{d}\right) = -\mu \frac{\partial}{\partial t}\left(\frac{I}{w}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{I}{w}\right) = -\sigma\left(\frac{V}{d}\right) - \epsilon \frac{\partial}{\partial t}\left(\frac{V}{d}\right)$$

or,

$$\boxed{\begin{aligned} \frac{\partial V}{\partial z} &= -\left(\frac{\mu d}{w}\right)\frac{\partial I}{\partial t} \\ \frac{\partial I}{\partial z} &= -\left(\frac{\sigma w}{d}\right)V - \left(\frac{\epsilon w}{d}\right)\frac{\partial V}{\partial t} \end{aligned}}$$

These equations are known as the <u>transmission line equations</u>. They characterize the wave propagation along the line in terms of voltage and current instead of in terms of the fields.

We now define three quantities as follows:

Inductance per unit length of the line, $\mathcal{L}$:

$$\mathcal{L} = \frac{\text{Magnetic flux per unit length}}{\text{current}} = \frac{B_y d}{H_y w} = \frac{\mu H_y d}{H_y w} = \frac{\mu d}{w}$$

Capacitance per unit length of the line, $\mathcal{C}$:

$$\mathcal{C} = \frac{\text{charge per unit length}}{\text{voltage}} = \frac{\rho_s w}{E_x d} = \frac{\epsilon E_x w}{E_x d} = \frac{\epsilon w}{d}$$

Conductance per unit length of the line, $\mathcal{G}$:

$$\mathcal{G} = \frac{\text{conduction current per unit length}}{\text{voltage}} = \frac{\sigma E_x w}{E_x d} = \frac{\sigma w}{d}$$

The transmission line equations then become

$$\boxed{\begin{aligned} \frac{\partial V}{\partial z} &= -\mathcal{L}\frac{\partial I}{\partial t} \qquad &(1) \\ \frac{\partial I}{\partial z} &= -\mathcal{G}V - \mathcal{C}\frac{\partial V}{\partial t} \qquad &(2) \end{aligned}}$$

Note that $\mathcal{L}\mathcal{C} = \mu\epsilon$ and $\mathcal{G}/\mathcal{C} = \sigma/\epsilon$.

We now derive the transmission equivalent circuit as follows:

Let us consider a section Δz of the line between z and $z+\Delta z$.

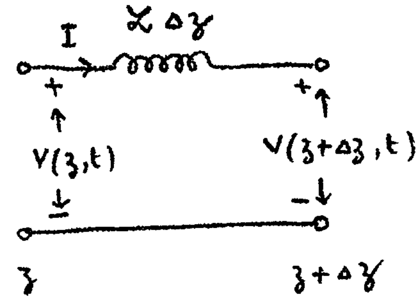
Then, we can write eq. (1) as

$$\lim_{\Delta z \to 0} \frac{V(z+\Delta z, t) - V(z,t)}{\Delta z} = -\mathcal{L} \frac{\partial I(z,t)}{\partial t}$$

or, for $\Delta z \to 0$,

$$V(z+\Delta z, t) - V(z,t) = -\mathcal{L}\Delta z \frac{\partial I(z,t)}{\partial t}$$

This equation can be represented by the circuit equivalent shown to the right.



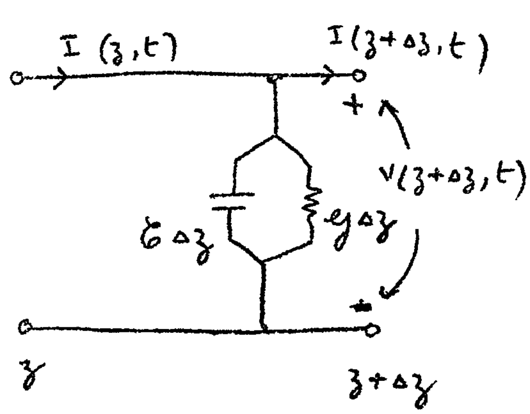
Similarly, we can write eq. (2) as

$$\lim_{\Delta z \to 0} \frac{I(z+\Delta z, t) - I(z,t)}{\Delta z} = \lim_{\Delta z \to 0}\left[-g\, V(z+\Delta z,t) - \mathcal{C} \frac{\partial V(z+\Delta z,t)}{\partial t}\right]$$

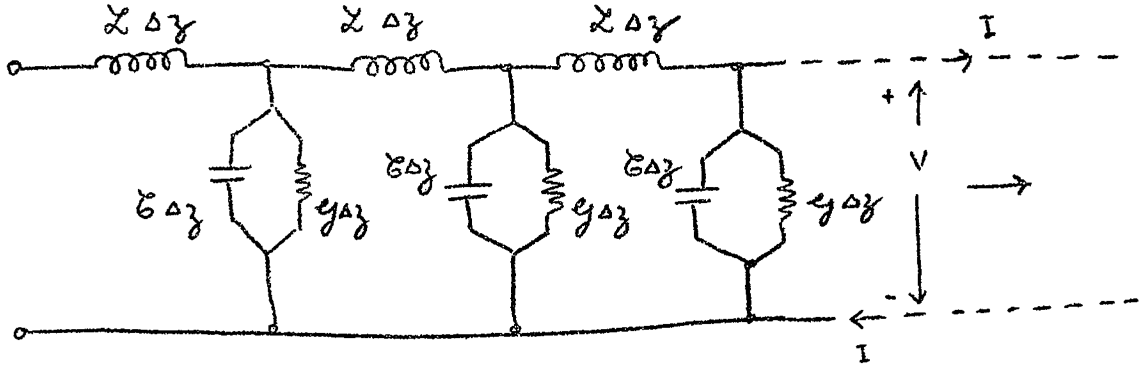
or, for $\Delta z \to 0$,

$$I(z+\Delta z, t) - I(z,t) = -g\Delta z\, V(z+\Delta z,t) - \mathcal{C}\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t}$$

This equation can be represented by the circuit equivalent shown to the right.



Combining the two equations and also considering a length l of the line, we obtain the following equivalent circuit for the line:



This is known as a "distributed circuit" as opposed to "lumped circuits", familiar in circuit theory. The distributed circuit notion arises from the fact that the inductance, capacitance and conductance are distributed uniformly and overlappingly along the line.

HOMEWORK PROBLEM (DUE 11/11/74)

By performing surface integration of the Poynting vector over the crosssectional area of the parallel plate transmission line, show that the power flow down the line is VI.

TRANSMISSION LINE WITH AN ARBITRARY CROSS SECTION:

In the previous lecture, we considered the parallel plate transmission line for which

$$\vec{E} = E_x(z,t)\,\vec{i}_x$$

$$\vec{H} = H_y(z,t)\,\vec{i}_y$$

and with the conductors lying in the planes $x = 0$ and $x = d$ so that the boundary conditions

$$\vec{i}_n \times \vec{E} = 0 \quad (\text{tangential component of } \vec{E} \text{ is zero})$$

$$\vec{i}_n \cdot \vec{B} = 0 \quad (\text{normal component of } \vec{B} \text{ is zero})$$

are satisfied, thereby leading to the situation of the uniform plane wave being guided by the conductors.

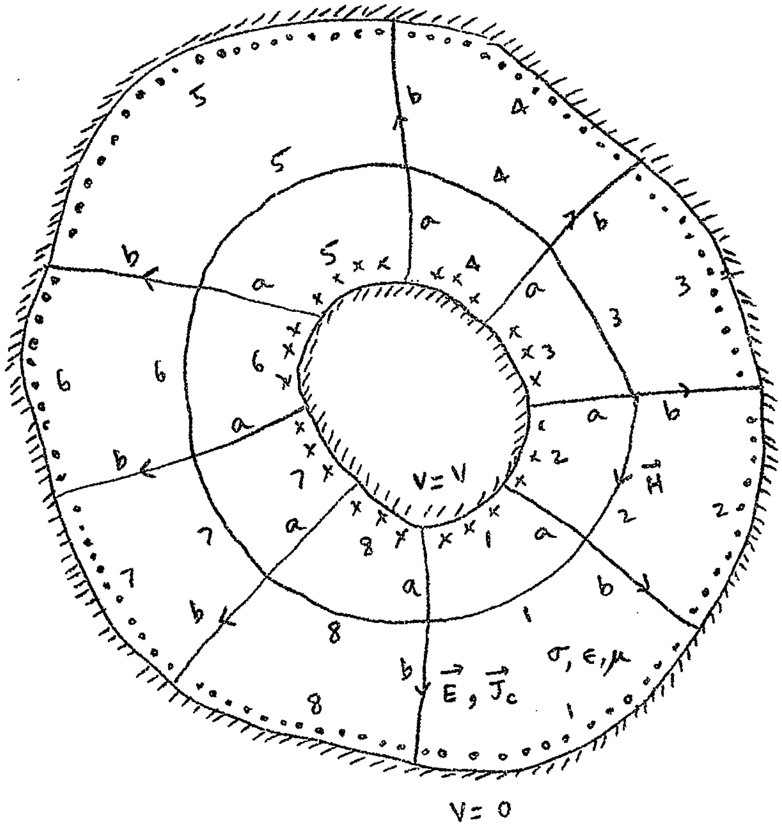
In the general case, the conductors of the transmission line have arbitrary crosssections and the fields between them are given by

$$\vec{E} = E_x(x,y,z,t)\,\vec{i}_x + E_y(x,y,z,t)\,\vec{i}_y$$

$$\vec{H} = H_x(x,y,z,t)\,\vec{i}_x + H_y(x,y,z,t)\,\vec{i}_y$$

Thus the fields are not uniform in x and y but they are entirely transverse to the direction of propagation, i.e., the axis of the transmission line. Hence they are known as "<u>transverse electromagnetic waves</u>," or <u>TEM waves</u>.

The fields must satisfy the boundary conditions at the conductors and must also be perpendicular to each other at each and every point between the two conductors. Hence by considering a crosssectional plane of the transmission line, one can sketch a field map in the following manner:



The field map is made up of curvillinear rectangles. By drawing a large number of field lines, we can make these rectangles so small that each one of them can be viewed as a parallel plate transmission line. In fact by choosing the spacings appropriately, we can make them a set of curvillinear squares.

From the field map, we can obtain the transmission line parameters $\mathscr{C}$, $\mathscr{L}$ and $\mathscr{G}$ as follows:

$$\mathscr{C} = \frac{Q}{V} = \frac{Q_1 + Q_2 + \cdots + Q_8}{V_a + V_b}$$

$$= \frac{Q_1}{V_a + V_b} + \frac{Q_2}{V_a + V_b} + \cdots + \frac{Q_8}{V_a + V_b}$$

$$= \frac{1}{\frac{V_a}{Q_1} + \frac{V_b}{Q_1}} + \frac{1}{\frac{V_a}{Q_2} + \frac{V_b}{Q_2}} + \cdots + \frac{1}{\frac{V_a}{Q_8} + \frac{V_b}{Q_8}}$$

$$= \frac{1}{\frac{1}{C_{1a}} + \frac{1}{C_{1b}}} + \frac{1}{\frac{1}{C_{2a}} + \frac{1}{C_{2b}}} + \cdots + \frac{1}{\frac{1}{C_{8a}} + \frac{1}{C_{8b}}}$$

If the map consists of curvilinear squares, then the capacitance for each square is $\epsilon \frac{w}{d} = \epsilon$ and $\mathscr{C} = 8 \frac{1}{\frac{1}{\epsilon} + \frac{1}{\epsilon}} = \frac{8}{2/\epsilon} = \epsilon \frac{8}{2}$.

In general, if we have m squares tangential to the conductors and n square normal to the conductors, then

$$\boxed{\mathscr{C} = \epsilon \frac{m}{n}}$$

$$\mathscr{L} = \frac{\Psi}{I} = \frac{\Psi_a + \Psi_b}{I_1 + I_2 + \cdots + I_8}$$

Ψ is magnetic flux

$$= \frac{\Psi_a}{I_1 + I_2 + \cdots + I_8} + \frac{\Psi_b}{I_1 + I_2 + \cdots + I_8}$$

$$= \frac{1}{\frac{I_1}{\Psi_a} + \frac{I_2}{\Psi_a} + \cdots + \frac{I_8}{\Psi_a}} + \frac{1}{\frac{I_1}{\Psi_b} + \frac{I_2}{\Psi_b} + \cdots + \frac{I_8}{\Psi_b}}$$

$$= \frac{1}{\frac{1}{\mathcal{L}_{1a}} + \frac{1}{\mathcal{L}_{2a}} + \cdots + \frac{1}{\mathcal{L}_{8a}}} + \frac{1}{\frac{1}{\mathcal{L}_{1b}} + \frac{1}{\mathcal{L}_{2b}} + \cdots + \frac{1}{\mathcal{L}_{8b}}}$$

If the map consists of curvillinear squares, then the inductance for each square is $\mu \frac{d}{w} = \mu$ and $\mathcal{L} = 2 \frac{1}{8/\mu} = \mu \frac{2}{8}$.

In general, if we have m squares tangential to the conductors and n squares normal to the conductors, then

$$\boxed{\mathcal{L} = \mu \frac{n}{m}}$$

$$\mathcal{G} = \frac{I_c}{V} = \frac{I_{c1} + I_{c2} + \cdots + I_{c8}}{V_a + V_b}$$

I_c is the conduction current from one conductor to the other.

$$= \frac{I_{c1}}{V_a + V_b} + \frac{I_{c2}}{V_a + V_b} + \cdots + \frac{I_{c8}}{V_a + V_b}$$

$$= \frac{1}{\frac{V_a}{I_{c1}} + \frac{V_b}{I_{c1}}} + \frac{1}{\frac{V_a}{I_{c2}} + \frac{V_b}{I_{c2}}} + \cdots + \frac{1}{\frac{V_a}{I_{c8}} + \frac{V_b}{I_{c8}}}$$

$$= \frac{1}{\frac{1}{\mathcal{G}_{1a}} + \frac{1}{\mathcal{G}_{1b}}} + \frac{1}{\frac{1}{\mathcal{G}_{2a}} + \frac{1}{\mathcal{G}_{2b}}} + \cdots + \frac{1}{\frac{1}{\mathcal{G}_{8a}} + \frac{1}{\mathcal{G}_{8b}}}.$$

In general,

$$\boxed{\mathcal{G} = \sigma \frac{m}{n}}$$

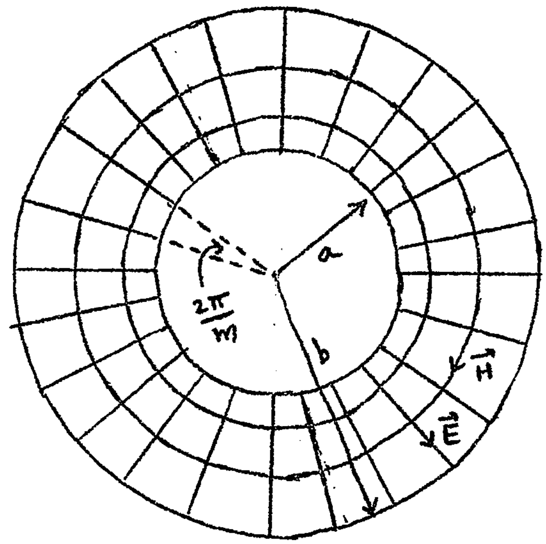
Note again that $\mathcal{L}\mathcal{C} = \mu\epsilon$ and $\frac{\mathcal{C}}{\mathcal{G}} = \frac{\epsilon}{\sigma}$.

EXAMPLE : Coaxial Cable :

Inner radius = a

Outer radius = b

Let the number of curvillinear squares in the angular direction, i.e., tangential to the conductors be m.



Then, to find the number of curvillinear squares in the radial direction, we note that the angle subtended at the center by adjacent pairs of electric field lines is $\frac{2\pi}{m}$. Hence at any arbitrary radius r, the side of the curvillinear square is equal to $r\left(\frac{2\pi}{m}\right)$. The number of squares in an infinitesimal distance dr in the radial direction is $\dfrac{dr}{r\left(\frac{2\pi}{m}\right)} = \dfrac{m}{2\pi}\dfrac{dr}{r}$.

The total number of squares in the radial direction from the inner conductor to the outer conductor is then given by

$$n = \int_{r=a}^{b} \frac{m}{2\pi}\frac{dr}{r} = \frac{m}{2\pi}\left[\ln r\right]_a^b = \frac{m}{2\pi}\ln\frac{b}{a}$$

and $\dfrac{m}{n} = \dfrac{2\pi}{\ln\frac{b}{a}}$.

Thus,

$$\boxed{\begin{aligned} \mathcal{C} &= \epsilon \frac{m}{n} = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \\ \mathcal{L} &= \mu \frac{n}{m} = \frac{\mu}{2\pi} \ln \frac{b}{a} \\ \mathcal{G} &= \sigma \frac{m}{n} = \frac{2\pi\sigma}{\ln \frac{b}{a}} \end{aligned}}$$

for a coaxial cable

Returning now to the transmission line equations, we have

$$\frac{\partial V}{\partial z} = -\mathcal{L} \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -\mathcal{G} V - \mathcal{C} \frac{\partial V}{\partial t}$$

For the sinusoidally time varying case, the corresponding equations for the phasor $\bar{V}$ and phasor $\bar{I}$ are

$$\frac{\partial \bar{V}}{\partial z} = -j\omega \mathcal{L} \bar{I}$$

$$\frac{\partial \bar{I}}{\partial z} = -\mathcal{G} \bar{V} - j\omega \mathcal{C} \bar{V} = -(\mathcal{G} + j\omega \mathcal{C}) \bar{V}$$

Combining the two equations, we obtain the wave equation as

$$\frac{\partial^2 \bar{V}}{\partial z^2} = -j\omega \mathcal{L} \frac{\partial \bar{I}}{\partial z} = j\omega \mathcal{L} (\mathcal{G} + j\omega \mathcal{C}) \bar{V}$$

$$= \bar{\gamma}^2 \bar{V}$$

where

$$\bar{\gamma} = \sqrt{j\omega\mathcal{L}(\mathcal{G}+j\omega\mathcal{C})}$$ is the propagation constant.

The solution for $\bar{V}$ is given by

$$\bar{V}(z) = \bar{A}\,e^{-\bar{\gamma}z} + \bar{B}\,e^{\bar{\gamma}z}$$

The corresponding solution for I is given by

$$\bar{I}(z) = -\frac{1}{j\omega\mathcal{L}}\frac{\partial\bar{V}}{\partial z} = -\frac{1}{j\omega\mathcal{L}}\left(-\bar{\gamma}\,\bar{A}\,e^{-\bar{\gamma}z} + \bar{\gamma}\,\bar{B}\,e^{\bar{\gamma}z}\right)$$

$$= \sqrt{\frac{\mathcal{G}+j\omega\mathcal{C}}{j\omega\mathcal{L}}}\left(\bar{A}\,e^{-\bar{\gamma}z} - \bar{B}\,e^{\bar{\gamma}z}\right)$$

$$= \frac{1}{\bar{Z}_0}\left(\bar{A}\,e^{-\bar{\gamma}z} - \bar{B}\,e^{\bar{\gamma}z}\right)$$

where

$$\bar{Z}_0 = \sqrt{\frac{j\omega\mathcal{L}}{\mathcal{G}+j\omega\mathcal{C}}}$$ is the <u>characteristic impedance</u> of the transmission line, analogous to the intrinsic impedance of the medium.

For a coaxial cable with perfect dielectric between the conductors, i.e., $\sigma = 0$, $\mathcal{G} = 0$

$$Z_0 = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = \sqrt{\frac{\mu}{2\pi}\ln\frac{b}{a}\Big/\frac{2\pi\epsilon}{\ln\frac{b}{a}}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\,\ln\frac{b}{a}.$$

For $\mu = \mu_0$, $\epsilon = 2.25\,\epsilon_0$, $b/a = 3.67$

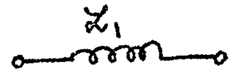
$$Z_0 = \frac{120\pi}{2\pi\sqrt{2.25}}\ln 3.67 = 40 \times 1.3 = 52\ \Omega.$$

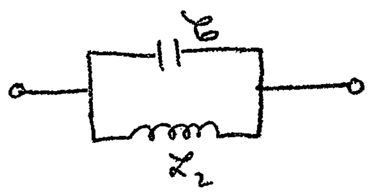
HOMEWORK PROBLEM (Due 11/13/74)

For a transmission line equivalent circuit having an impedance $\bar{Z}\,dz$ in the series branch and an admittance $\bar{Y}\,dz$ in the shunt branch,

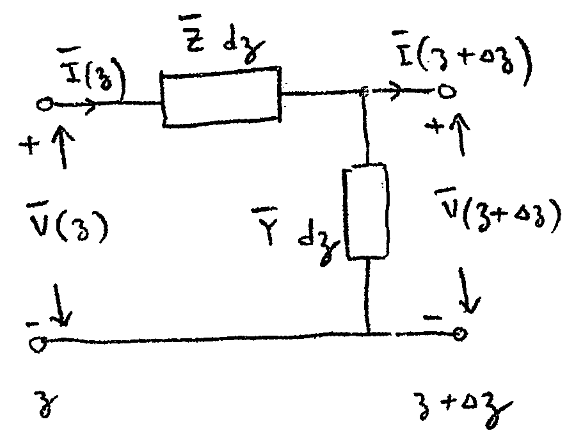
a) Write the transmission line equations.

b) Show that the propagation constant $\bar{\gamma} = \sqrt{\bar{Z}\bar{Y}}$ and that the characteristic impedance $\bar{Z}_0 = \sqrt{\bar{Z}/\bar{Y}}$.

c) If $\bar{Z}$ is made up of 

and $\bar{Y}$ is made up of 

, find $\bar{\gamma}$ and discuss the propagation characteristics.



SHORT CIRCUITED TRANSMISSION LINE:

We found in the previous lecture that the solutions for voltage and current on a transmission line are given by

$$\bar{V}(z) = \bar{A} e^{-\bar{\gamma} z} + \bar{B} e^{\bar{\gamma} z}$$

$$\bar{I}(z) = \underbrace{\frac{\bar{A}}{\bar{Z}_0} e^{-\bar{\gamma} z}}_{(+)\text{ wave}} - \underbrace{\frac{\bar{B}}{\bar{Z}_0} e^{\bar{\gamma} z}}_{(-)\text{ wave}}$$

where $\bar{\gamma} = \sqrt{j\omega \mathcal{L}(\mathcal{G} + j\omega \mathcal{C})}$ is the propagation constant,

and $\bar{Z}_0 = \sqrt{\dfrac{j\omega\mathcal{L}}{\mathcal{G} + j\omega\mathcal{C}}}$ is the characteristic impedance.

Let us now consider for simplicity a lossless line, i.e., one with a perfect dielectric ($\sigma = 0$) between the two conductors. Then $\mathcal{G} = 0$ and

$$\bar{\gamma} = \alpha + j\beta = \sqrt{j\omega\mathcal{L} \cdot j\omega\mathcal{C}} = j\omega\sqrt{\mathcal{L}\mathcal{C}}$$

$$\beta = \omega\sqrt{\mathcal{L}\mathcal{C}} \quad \text{phase constant} \quad (= \omega\sqrt{\mu\epsilon} \text{ since } \mathcal{L}\mathcal{C} = \mu\epsilon)$$

$$\bar{Z}_0 = \sqrt{j\omega\mathcal{L}/j\omega\mathcal{C}} = \sqrt{\mathcal{L}/\mathcal{C}} \quad \text{is purely real}$$

We can then write

$$\bar{V}(z) = \bar{A} e^{-j\beta z} + \bar{B} e^{j\beta z}$$

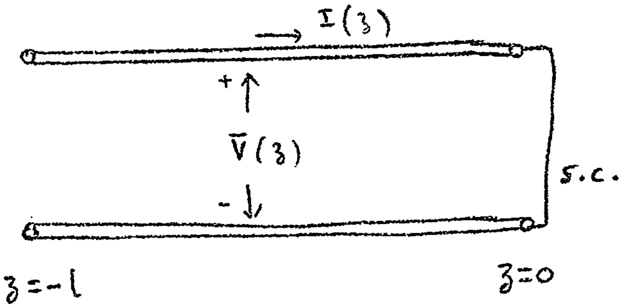
$$\bar{I}(z) = \frac{1}{Z_0}\left[\bar{A} e^{-j\beta z} - \bar{B} e^{j\beta z}\right]$$

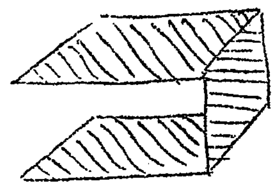
Actually,

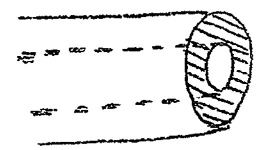
$$V(z,t) = \mathrm{Re}\left[\bar{V}(z) \cdot e^{j\omega t}\right] = A\cos(\omega t - \beta z + \theta) + B\cos(\omega t + \beta z + \phi)$$

$$I(z,t) = \mathrm{Re}\left[\bar{I}(z) \cdot e^{j\omega t}\right] = \frac{1}{Z_0}\left[A\cos(\omega t - \beta z + \theta) - B\cos(\omega t + \beta z + \phi)\right]$$

Let us now short circuit the line at the far end and drive it at the left end by a voltage generator of frequency ω.







Then a <u>boundary condition</u> must be satisfied at the short circuit. This boundary condition is that the tangential electric field at the surface of the conductor making up the short circuit must be zero. This in turn means that the voltage across the short circuit must be zero. Thus

$$\bar{V}(0) = 0 \qquad \text{B.C.}$$

Applying this to the general solutions for $\bar{V}$ and $\bar{I}$, we obtain

$$0 = \bar{A}\, e^{-j\beta(0)} + \bar{B}\, e^{j\beta(0)} \qquad \text{or} \qquad \bar{A} + \bar{B} = 0$$

$$\bar{B} = -\bar{A}, \text{ and}$$

$$\bar{V}(z) = \bar{A}\, e^{-j\beta z} - \bar{A}\, e^{j\beta z} = -2j\bar{A} \sin\beta z$$

$$\bar{I}(z) = \frac{1}{Z_0}\left[\bar{A}\, e^{-j\beta z} + \bar{A}\, e^{j\beta z}\right] = \frac{2\bar{A}}{Z_0} \cos\beta z$$

Actually,

$$V(z,t) = Re\left[\bar{V}(z)\cdot e^{j\omega t}\right] = Re\left[+2\, e^{-j\frac{\pi}{2}} A e^{j\theta} \sin\beta z \cdot e^{j\omega t}\right]$$

$$= 2A \sin\beta z \sin(\omega t + \theta)$$

$$I(z,t) = Re\left[\bar{I}(z)\cdot e^{j\omega t}\right] = Re\left[2\frac{A}{Z_0} e^{j\theta} \cos\beta z \cdot e^{j\omega t}\right]$$

$$= \frac{2A}{Z_0} \cos\beta z \cos(\omega t + \theta)$$

These solutions correspond to complete standing waves, resulting from (+) and (−) waves of equal amplitude, due to perfect reflection from the short circuit. There is no time average power flow down the line since

$$P(z,t) = V(z,t)\cdot I(z,t) = \frac{4A^2}{Z_0}\sin\beta z\cos\beta z\cdot\sin(\omega t+\theta)\cdot\cos(\omega t+\theta)$$

$$= \frac{A^2}{Z_0}\sin 2\beta z\cdot\sin 2(\omega t+\theta)$$

Time average power $\langle P\rangle = \frac{1}{T}\int_0^T P\,dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P\,dt$

$$= \frac{\omega}{2\pi}\frac{A^2}{Z_0}\sin 2\beta z\int_0^{\frac{2\pi}{\omega}}\sin 2(\omega t+\theta)\,dt = 0$$

This is to be expected since the short circuit cannot absorb any power.

STANDING WAVE PATTERNS

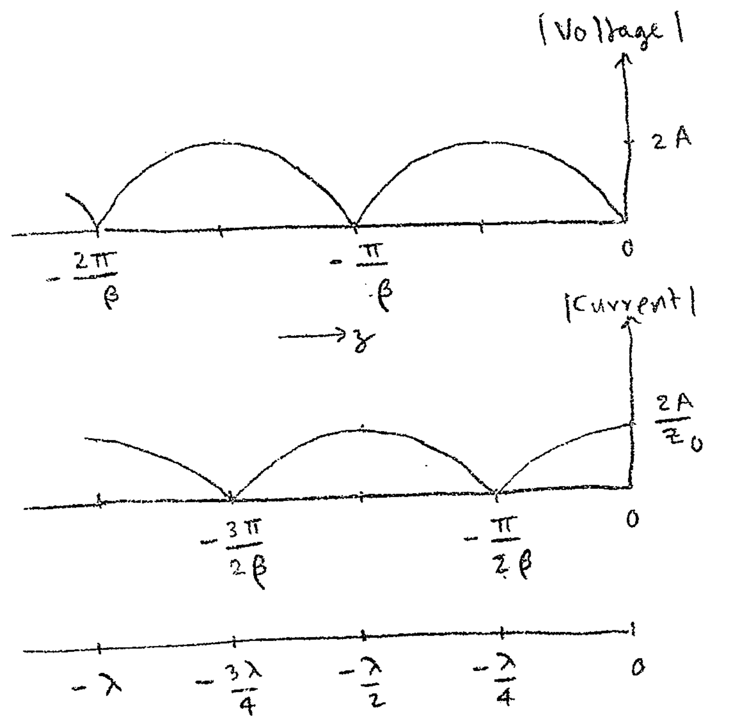
are simply sketches of variations of line voltage amplitude and line current amplitude along the line. Thus

$$|\bar V(z)| = |-2j\bar A\sin\beta z|$$

$$= 2A|\sin\beta z|$$

$$|\bar I(z)| = \left|\frac{2\bar A}{Z_0}\cos\beta z\right|$$

$$= \frac{2A}{Z_0}|\cos\beta z|$$



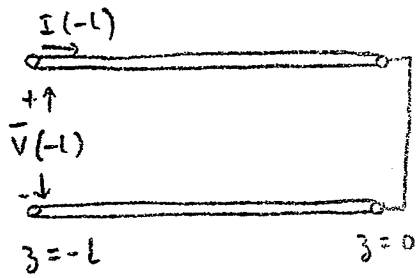
$$\beta = \frac{2\pi}{\lambda}.$$

$$\frac{\pi}{2\beta} = \frac{\pi}{4\pi/\lambda} = \frac{\lambda}{4}.$$

INPUT IMPEDANCE OF THE SHORT-CIRCUITED LINE

Let us look at the input impedance of the line, i.e., the impedance viewed by the voltage generator driving the line. This is given by

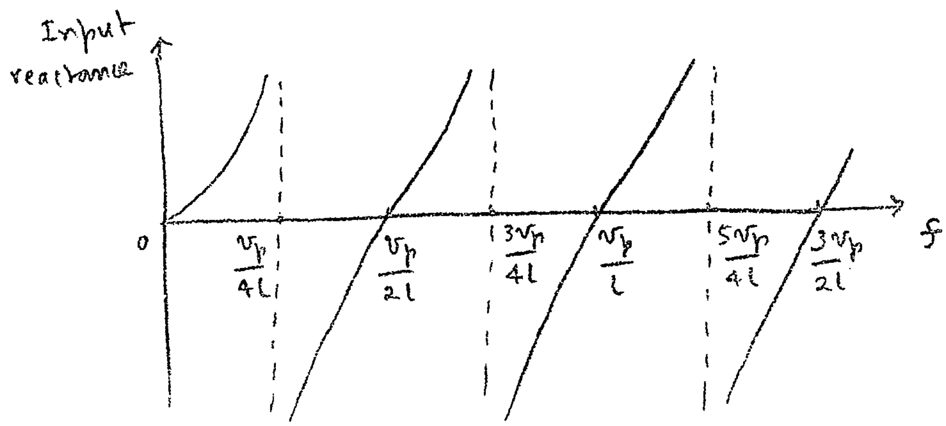
$$\bar{Z}_{in} = \frac{\bar{V}(-l)}{\bar{I}(-l)} = \frac{-2j\bar{A}\sin\beta(-l)}{\frac{2\bar{A}}{Z_0}\cos\beta(-l)}$$



$$\boxed{\bar{Z}_{in} = j Z_0 \tan\beta l}$$

$$= j Z_0 \tan\frac{\omega}{v_p} l \qquad \text{where } v_p = \frac{1}{\sqrt{\mathcal{L}\mathcal{C}}} \text{ is the phase velocity.}$$

As the frequency is varied from a low value upwards, the input impedance changes from inductive to capacitive and so on. The variation of the input reactance with frequency is sketched below.



Thus as the generator frequency is varied continuously, the current drawn from the generator undergoes minima and maxima (corresponding to the infinities and zeros in the input reactance). The difference between consecutive frequencies for which this occurs is $\frac{v_p}{4l}$. This phenomenon can be employed in practice to locate a short circuit occured somewhere in the line.

EXAMPLE: To determine the location of a short circuit in an air-insulated parallel wire line, a voltage generator of variable frequency is connected at its input and the current drawn from the generator is monitored as the frequency is varied. It is found that the current reaches a minimum at 100.02 MHz and then a maximum at 100.05 MHz (The frequency sweep was started at 100 MHz).

If l is the distance of the short circuit from the generator, we then have

$$\frac{v_p}{4l} = 100.05 - 100.02 = 0.03 \text{ MHz} = 3 \times 10^4$$

Since $v_p = 3\times10^8$ in view of the air dielectric (we assume a lossless line),

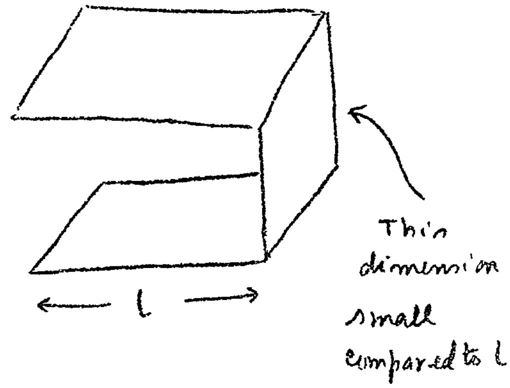
$$l = \frac{3\times10^8}{4\times3\times10^4} = \frac{10^4}{4} \text{ m} = 2.5 \text{ km}$$

HOMEWORK PROBLEM (Due 11/15/74)

Solve the above example from a consideration of what happens to the standing wave pattern for the voltage as the frequency is varied, i.e., look at the sketches of the standing wave patterns between the generator and the short circuit for both frequencies and deduce the number of wavelengths (or half wavelengths or quarter wavelengths) at one of the two frequencies, which then gives the location of the short circuit.

STRAY CAPACITANCE OF AN INDUCTOR:

Let us consider a single turn inductor →



This structure is in fact a short-circuited transmission line at any arbitrary frequency. But for low frequencies, it behaves like an inductor.

How low a frequency?

To find out, we look at the input impedance of the short circuited line and note that

$$\bar{Z}_{in} = j Z_0 \tan \beta l$$

$$\approx j Z_0 \beta l \qquad \text{for } \beta l \ll 1$$

$$= j \sqrt{\frac{\mathcal{L}}{\mathcal{C}}}\ \omega \sqrt{\mathcal{L}\mathcal{C}}\ l \qquad \frac{2\pi}{\lambda} l \ll 1$$

$$= j \omega \mathcal{L} l \qquad l \ll \frac{\lambda}{2\pi} \quad \text{or} \quad l \ll \frac{v_p}{2\pi f}$$

$$= \text{inductor } \mathcal{L} l \qquad f \ll \frac{2\pi l}{v_p}$$

Thus for $l \ll \frac{\lambda}{2\pi}$, the above structure behaves like a single inductor.

If we relax this assumption slightly, we can no longer approximate $\tan \beta l$ by βl and we have to include one more term in the infinite series expansion for $\tan \beta l$.

$$\tan \beta l = \beta l + \frac{1}{3}(\beta l)^3 + \frac{2}{15}(\beta l)^5 + \cdots$$

Thus

$$\bar{Z}_{in} = jZ_0 \tan \beta l$$

$$\approx jZ_0 \left(\beta l + \frac{1}{3} \beta^3 l^3 \right)$$

$$= j\sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \left(\omega\sqrt{\mathcal{L}\mathcal{C}}\, l + \frac{1}{3} \omega^3 \mathcal{L}^{3/2} \mathcal{C}^{3/2} l^3 \right)$$

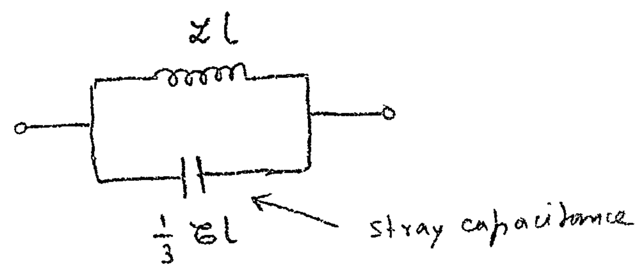
$$= j \left(\omega \mathcal{L} l + \frac{1}{3} \omega^3 \mathcal{L}^2 \mathcal{C} l^3 \right)$$

$$= j\omega \mathcal{L} l \left(1 + \frac{1}{3} \omega^2 \mathcal{L}\mathcal{C} l^2 \right)$$

$$\bar{Y}_{in} = \frac{1}{j\omega \mathcal{L} l} \left(1 + \frac{1}{3} \omega^2 \mathcal{L}\mathcal{C} l^2 \right)^{-1}$$

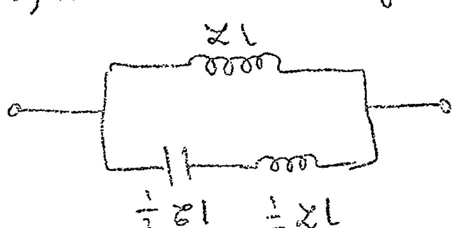
$$\approx \frac{1}{j\omega \mathcal{L} l} \left(1 - \frac{1}{3} \omega^2 \mathcal{L}\mathcal{C} l^2 \right)$$

$$= \frac{1}{j\omega \mathcal{L} l} + j\frac{1}{3}\omega \mathcal{C} l \quad =$$



Thus at frequencies somewhat above those for which $f \ll \frac{2\pi l}{v_p}$, a stray capacitance $\frac{1}{3}\mathcal{C} l$ has to be included in parallel with the inductor value.

If the frequency is increased still further, the equivalent circuit becomes



and so on.

BOUNDARY CONDITIONS AT A DIELECTRIC DISCONTINUITY:

Medium 1

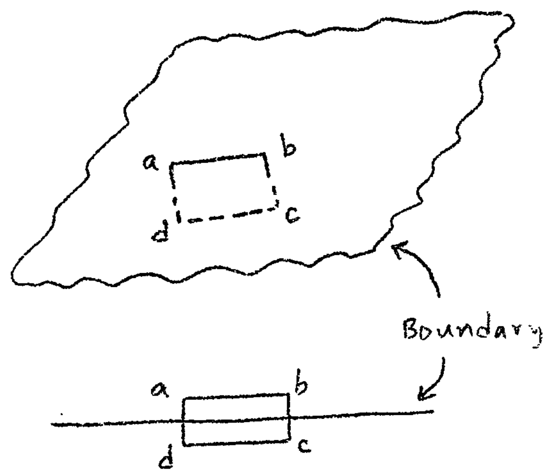
ϵ_1, μ_1

Boundary

Medium 2

ϵ_2, μ_2

① $\oint_c \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_s \vec{B} \cdot \vec{ds}$



Applying it to the rectangular path abcda, we have

$$\int_a^b \vec{E} \cdot \vec{dl} + \int_b^c \vec{E} \cdot \vec{dl} + \int_c^d \vec{E} \cdot \vec{dl} + \int_d^a \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_{abcd} \vec{B} \cdot \vec{ds}$$

In the limit that ad and bc $\to 0$, we obtain

$$E_{ab}\,(ab) + E_{cd}\,(cd) = 0$$

$$E_{ab}\,(ab) - E_{dc}\,(cd) = 0 \qquad \text{or} \qquad E_{ab} = E_{dc}$$

Since this is true for any orientation of the rectangle, it follows that

B.C. I

$$\boxed{E_{t1} = E_{t2} \qquad \text{or} \qquad \vec{i}_n \times (\vec{E}_1 - \vec{E}_2) = 0}$$

The tangential component of $\vec{E}$ is continuous.

② $$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} + \frac{d}{dt}\int_S \vec{D}\cdot d\vec{s}$$

Applying it to the rectangular path abcda and letting ad and bc $\to 0$ as in the case of ①, we obtain

$H_{ab}\,(ab) + H_{cd}\,(cd) = 0$ (There is no surface current on the boundary)

$H_{ab}\,(ab) - H_{dc}\,(cd) = 0$ or $H_{ab} = H_{dc}$

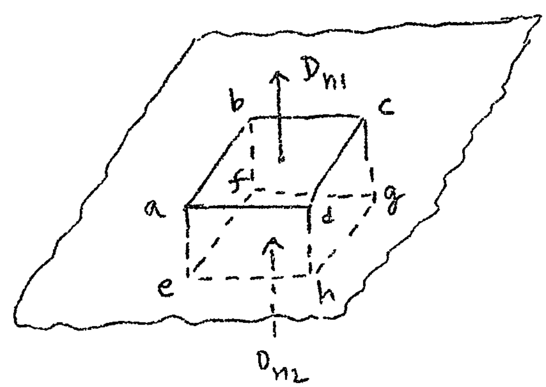
B.C. II

$H_{t1} = H_{t2}$ or $\vec{i}_n \times (\vec{H}_1 - \vec{H}_2) = 0$

The tangential component of $\vec{H}$ is continuous.

③ $$\oint_S \vec{D}\cdot d\vec{s} = \int_V \rho\, dv$$

Applying it to the rectangular box abcdefgh, we have



$$\int_{abcd} \vec{D}\cdot d\vec{s} + \int_{\substack{\text{side}\\\text{surfaces}}} \vec{D}\cdot d\vec{s} + \int_{efgh} \vec{D}\cdot d\vec{s} = \int_{\substack{\text{Volume}\\\text{of the box}}} \rho\, dv$$

In the limit that the side surfaces $\to 0$, we obtain

$D_{n1}\,(abcd) - D_{n2}\,(efgh) = 0$ (There is no surface charge on the boundary)

B.C. III

$D_{n1} = D_{n2}$ or $\vec{i}_n \cdot (\vec{D}_1 - \vec{D}_2) = 0$

The normal component of $\vec{D}$ is continuous.

④ $\oint_S \vec{B} \cdot d\vec{s} = 0$

Applying it to the rectangular box abcdefgh and letting the side surfaces $\to 0$, we obtain

$B_{n1}(abcd) - B_{n2}(efgh) = 0$

B.C. IV

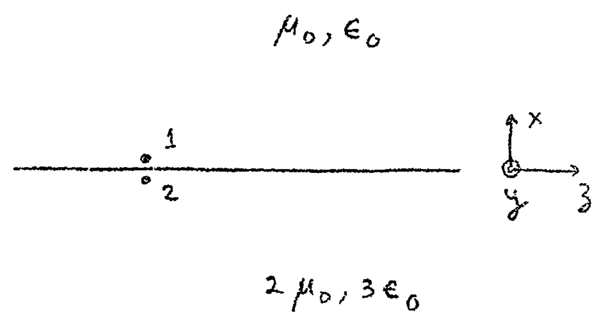
$$\boxed{B_{n1} = B_{n2} \quad \text{or} \quad \vec{i}_n \cdot (\vec{B}_1 - \vec{B}_2) = 0}$$

The normal component of $\vec{B}$ is continuous.

EXAMPLE: Given

$\vec{E}_1 = E_0(3\vec{i}_x + \vec{i}_z)$

$\vec{H}_1 = H_0(2\vec{i}_y)$



Then

$D_{2x} = D_{1x} = \epsilon_0(3E_0) = 3\epsilon_0 E_0$

$E_{2x} = \dfrac{D_{2x}}{3\epsilon_0} = \dfrac{3\epsilon_0 E_0}{3\epsilon_0} = E_0$

$E_{2y} = E_{1y} = 0$

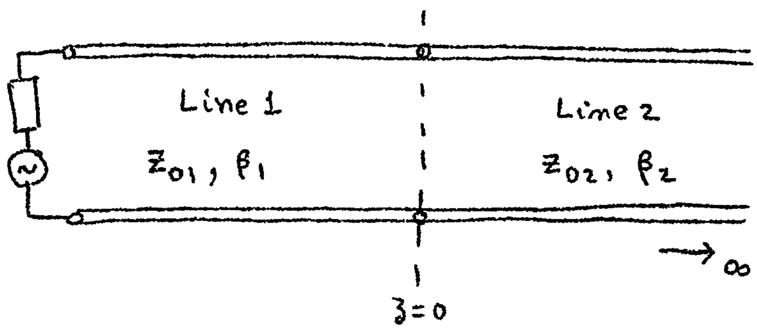
$E_{2z} = E_{1z} = E_0$ $\quad\quad \therefore \vec{E}_2 = E_0(\vec{i}_x + \vec{i}_z)$

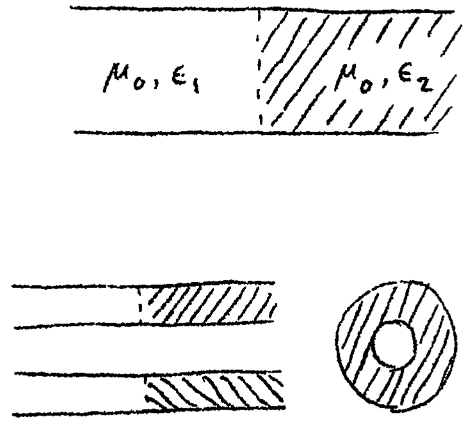
$B_{2x} = B_{1x} = 0, \quad H_{2x} = 0$

$H_{2y} = H_{1y} = 2H_0$

$H_{2z} = H_{1z} = 0$ $\quad\quad \therefore \vec{H}_2 = H_0(2\vec{i}_y)$

TRANSMISSION-LINE DISCONTINUITY:





Assume line 2 to be infinitely long so that there is no (−) wave in that line. Then from the general solutions to the line voltage and current, we have

$$\left.\begin{aligned} \bar{V}_1(z) &= \bar{V}_1^+ e^{-j\beta_1 z} + \bar{V}_1^- e^{j\beta_1 z} \\ \bar{I}_1(z) &= \frac{1}{Z_{01}}\left[\bar{V}_1^+ e^{-j\beta_1 z} - \bar{V}_1^- e^{j\beta_1 z}\right] \end{aligned}\right\} \text{ Line 1} \qquad \begin{array}{l}\text{incident wave} \\ \text{+ reflected wave}\end{array}$$

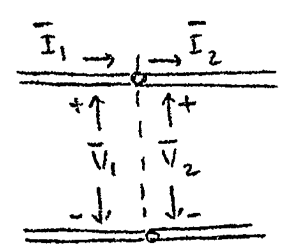
$$\left.\begin{aligned} \bar{V}_2(z) &= \bar{V}_2^+ e^{-j\beta_2 z} \\ \bar{I}_2(z) &= \frac{1}{Z_{02}} \bar{V}_2^+ e^{-j\beta_2 z} \end{aligned}\right\} \text{ Line 2} \qquad \text{transmitted wave}$$

where $\bar{V}_1^+$, $\bar{V}_1^-$ and $\bar{V}_2^+$ are constants to be determined from the boundary conditions.

<u>At the junction</u> between the two lines, the boundary conditions require

$$E_{t1} = E_{t2} \longrightarrow \bar{V}_1 = \bar{V}_2$$

$$H_{t1} = H_{t2} \longrightarrow \bar{I}_1 = \bar{I}_2$$



Thus $\quad \bar{V}_1(z=0-) = \bar{V}_2(z=0+)$

$$\bar{V}_1^+ + \bar{V}_1^- = \bar{V}_2^+ \quad \text{——} \quad (1) \qquad \frac{1}{Z_{02}}(\bar{V}_1^+ + \bar{V}_1^-) = \frac{1}{Z_0}\bar{V}_2^+$$

$$\bar{I}_1(z=0-) = \bar{I}_2(z=0+)$$

$$\frac{1}{Z_{01}}(\bar{V}_1^+ - \bar{V}_1^-) = \frac{1}{Z_{02}}\bar{V}_2^+ \quad \text{—} \quad (2)$$

From (1) and (2), we have

$$\bar{V}_1^+\left(\frac{1}{Z_{02}} - \frac{1}{Z_{01}}\right) + \bar{V}_1^-\left(\frac{1}{Z_{02}} + \frac{1}{Z_{01}}\right) = 0$$

or $$\bar{V}_1^- \frac{Z_{01}+Z_{02}}{Z_{01}Z_{02}} = -\bar{V}_1^+ \frac{Z_{01}-Z_{02}}{Z_{01}Z_{02}}$$

$$\bar{V}_1^- = \bar{V}_1^+ \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}$$

We now define the <u>voltage reflection coefficient at the junction</u> as

$$\boxed{\Gamma_V = \frac{\bar{V}_1^-}{\bar{V}_1^+} = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}}$$

and the <u>voltage transmission coefficient at the junction</u> as

$$\tau_V = \frac{\bar{V}_2^+}{\bar{V}_1^+} = \frac{\bar{V}_1^+ + \bar{V}_1^-}{V_1^+} = 1 + \frac{\bar{V}_1^-}{\bar{V}_1^+} = 1 + \Gamma_V = \frac{2Z_{02}}{Z_{02}+Z_{01}}$$

and the <u>current transmission coefficient at the junction</u> as

$$\tau_I = \frac{I_2^+}{\bar{I}_1^+} = \frac{\bar{I}_1^+ + \bar{I}_1^-}{I_1^+} = 1 + \frac{I_1^-}{I_1^+} = 1 + \Gamma_I$$

where Γ_I is the current reflection coefficient at the junction given by

$$\Gamma_I = \frac{\bar{I}_1^-}{\bar{I}_1^+} = \frac{-\bar{V}_1^-/Z_{01}}{\bar{V}_1^+/Z_{01}} = -\frac{\bar{V}_1^-}{\bar{V}_1^+} = -\Gamma_V .$$

Thus

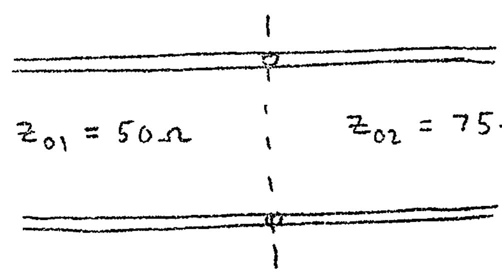
$$\boxed{\Gamma_V = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}}, \quad \Gamma_I = -\Gamma_V, \quad \tau_V = 1+\Gamma_V, \quad \tau_I = 1-\Gamma_V}$$

Note that for $Z_{02} = Z_{01}$, $\Gamma_V = 0$, $\Gamma_I = 0$, $\tau_V = \tau_I = 1$, i.e., no reflection and complete transmission.

EXAMPLE:

For $Z_{01} = 50\,\Omega$ and

$Z_{02} = 75\,\Omega$



$$\Gamma_V = \frac{75-50}{75+50} = \frac{25}{125} = \frac{1}{5} \longrightarrow \bar{V}_1^- = \frac{1}{5}\bar{V}_1^+$$

$$\Gamma_I = -\Gamma_V = -\frac{1}{5} \longrightarrow \bar{I}_1^- = -\frac{1}{5}\bar{I}_1^+$$

$$\tau_V = 1+\Gamma_V = 1+\frac{1}{5} = \frac{6}{5} \longrightarrow \bar{V}_2^+ = \frac{6}{5}\bar{V}_1^+$$

$$\tau_I = 1-\Gamma_V = 1-\frac{1}{5} = \frac{4}{5} \qquad I_2^+ = \frac{4}{5}\bar{I}_1^+$$

If the incident power on the junction is P_i watts, then

reflected power $= \Gamma_V \Gamma_I P_i = -\frac{1}{25} P_i$ (− sign simply signifies power flow in the $-z$ direction)

transmitted power $= \tau_V \tau_I P_i = \frac{24}{25} P_i$. Power balance is satisfied.

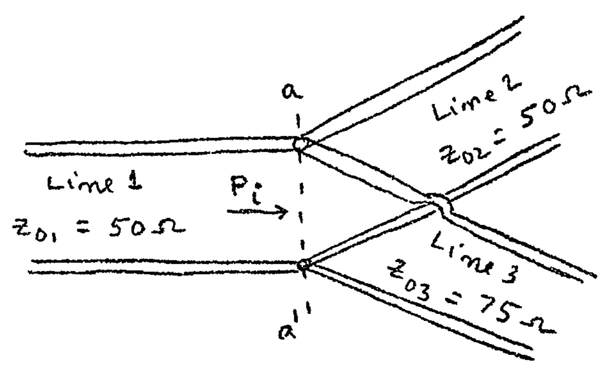
HOMEWORK PROBLEM (Due 11/18/74)

In the transmission line system shown below, a power P_i watts is incident on the junction a a' from the left. Find

a) the power reflected back into line 1

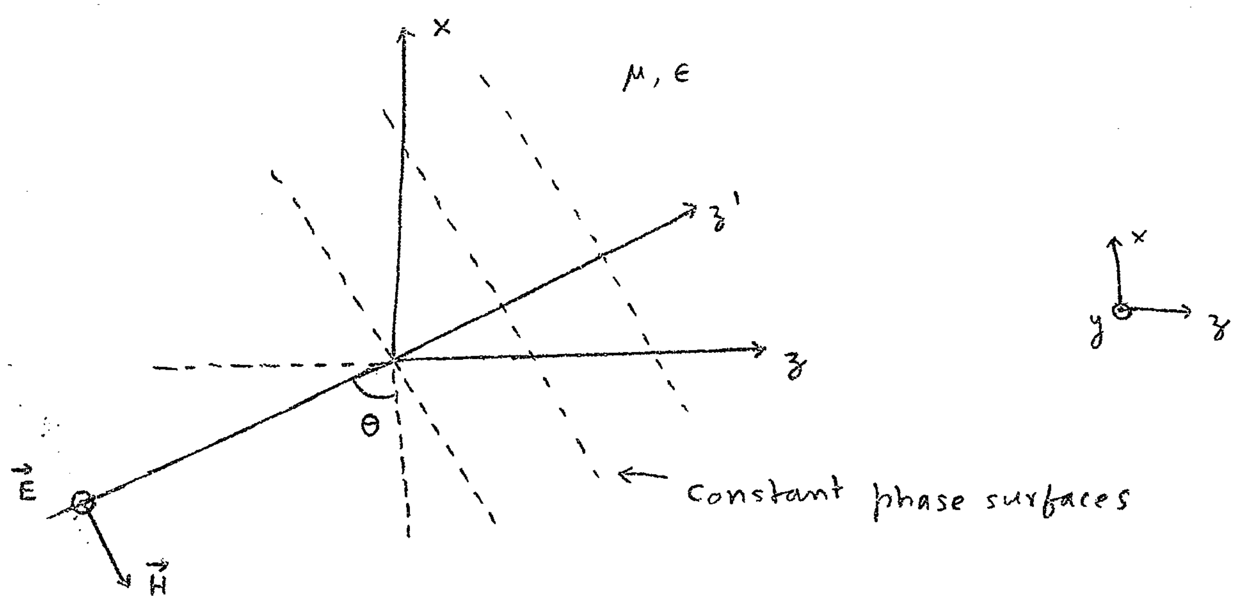
b) the power transmitted into line 2, and

c) the power transmitted into line 3.



Note that lines 2 and 3 are in parallel and hence division of transmitted current takes place whereas the transmitted voltages into the two lines are the same.

UNIFORM PLANE WAVE PROPAGATION IN AN ARBITRARY DIRECTION:



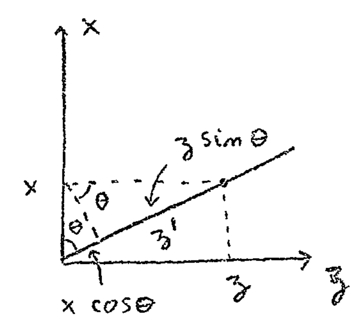
Let us consider a uniform plane wave having its electric field entirely in the y direction and propagating in the z' direction. Then

$$\vec{E} = E_0 \cos(\omega t - \beta z')\,\vec{i}_y \qquad \text{where} \quad \beta = \omega\sqrt{\mu\epsilon}$$

$$= E_0 \cos[\omega t - \beta(x\cos\theta + z\sin\theta)]\,\vec{i}_y$$

$$= E_0 \cos(\omega t - \beta\cos\theta \cdot x - \beta\sin\theta \cdot z)\,\vec{i}_y$$

$$= E_0 \cos(\omega t - \beta_x x - \beta_z z)\,\vec{i}_y$$

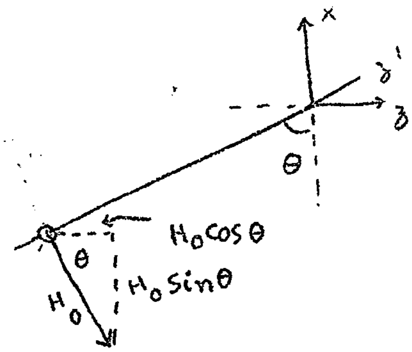


where $\beta_x = \beta\cos\theta$ and $\beta_z = \beta\sin\theta$ are the phase constants, i.e., the rates of change of phase along the x and z directions, respectively. Note that

$$\beta_x^2 + \beta_z^2 = \beta^2\cos^2\theta + \beta^2\sin^2\theta = \beta^2.$$

$$\vec{H} = \vec{H}_0 \cos(\omega t - \beta z')$$

$$= H_0(-\sin\theta\, \vec{i}_x + \cos\theta\, \vec{i}_z) \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta)$$



Since $\dfrac{|\vec{E}|}{|\vec{H}|} = \eta = \sqrt{\dfrac{\mu}{\epsilon}}$,

$$H_0^2(\sin^2\theta + \cos^2\theta) = \frac{E_0^2}{\eta}$$

$$H_0 = \frac{E_0}{\eta}.$$

$$\vec{H} = \frac{E_0}{\eta}(-\sin\theta\, \vec{i}_x + \cos\theta\, \vec{i}_z) \cdot \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta).$$

$$= -\frac{E_0}{\eta}\sin\theta \cdot \cos(\omega t - \beta_x x - \beta_z z)\vec{i}_x + \frac{E_0 \cos\theta}{\eta} \cdot \cos(\omega t - \beta_x x - \beta_z z)\, \vec{i}_z$$

Generalizing to a completly arbitrary direction in three dimensions x, y and z, we can write the electric and magnetic fields of such a wave as

$$\vec{E} = \vec{E}_0 \cos(\omega t - \beta_x x - \beta_y y - \beta_z z)$$

$$= \vec{E}_0 \cos\left[\omega t - (\beta_x \vec{i}_x + \beta_y \vec{i}_y + \beta_z \vec{i}_z)\cdot(x\vec{i}_x + y\vec{i}_y + z\vec{i}_z)\right]$$

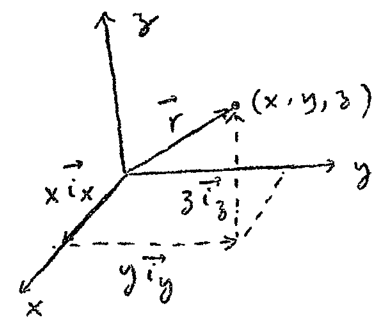
$$\boxed{\vec{E} = \vec{E}_0 \cos\left[\omega t - \vec{\beta}\cdot\vec{r}\right]}$$

where $\vec{\beta} = \beta_x \vec{i}_x + \beta_y \vec{i}_y + \beta_z \vec{i}_z$ is the propagation vector having magnitude equal to $\beta = \omega\sqrt{\mu\epsilon}$ and direction which is the direction of propagation. Its components along the coordinate axes are the phase constants along those respective axes.

The vector $\vec{r} = x\,\vec{i}_x + y\,\vec{i}_y + z\,\vec{i}_z$ is known as the position vector.

It is simply the vector drawn from the origin to the point (x, y, z).

$$\boxed{\vec{H} = \vec{H}_0 \cos(\omega t - \vec{\beta}\cdot\vec{r})}$$



Since $\vec{E}$, $\vec{H}$ and the direction of propagation are mutually perpendicular to each other, we have

$$\boxed{\begin{aligned} \vec{E}_0 \cdot \vec{\beta} &= 0 \\ \vec{H}_0 \cdot \vec{\beta} &= 0 \\ \vec{E}_0 \cdot \vec{H}_0 &= 0 \end{aligned}} \qquad \begin{aligned} &\vec{E}_0 \perp \vec{\beta} \\ &\vec{H}_0 \perp \vec{\beta} \\ &\vec{E}_0 \perp \vec{H}_0 \end{aligned}$$

Also,

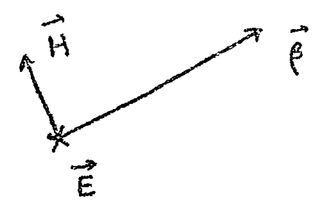
$$\boxed{\frac{|\vec{E}|}{|\vec{H}|} = \frac{|\vec{E}_0|}{|\vec{H}_0|} = \eta = \sqrt{\frac{\mu}{\epsilon}}}$$

$\vec{E} \times \vec{H}$ should be directed along the direction of propagation, i.e., $\vec{\beta}$.

We can combine these facts into a single equation as

$$\vec{H} = \frac{\vec{i}_\beta \times \vec{E}}{\eta} = \frac{\vec{i}_\beta \times \vec{E}}{\sqrt{\mu/\epsilon}} = \frac{\omega\sqrt{\mu\epsilon}\ \vec{i}_\beta \times \vec{E}}{\omega\mu}$$

$$\boxed{\vec{H} = \frac{1}{\omega\mu}\,\vec{\beta} \times \vec{E}}$$



EXAMPLE: Let us consider a 12 MHz uniform plane wave propagating in free space and given by the electric field vector

$$\vec{E} = 5(-\sqrt{3}\,\vec{i}_x + \vec{i}_y)\cos\left[24\pi\times10^6 t - 0.02\pi(\sqrt{3}x + 3y + 2z)\right] \text{ V/m.}$$

Then comparing with the general expression for $\vec{E}$, we have

$$\vec{\beta}\cdot\vec{r} = 0.02\pi(\sqrt{3}x + 3y + 2z) = 0.02\pi(\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)\cdot(x\vec{i}_x + y\vec{i}_y + z\vec{i}_z)$$

$$\therefore \vec{\beta} = 0.02\pi(\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)$$

Direction of propagation is along the unit vector $\dfrac{\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z}{\sqrt{3+9+4}}$

or $\quad \dfrac{\sqrt{3}}{4}\vec{i}_x + \dfrac{3}{4}\vec{i}_y + \dfrac{1}{2}\vec{i}_z.$

$$\beta = |\vec{\beta}| = 0.02\pi\,|\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z| = 0.08\pi$$

$$= \frac{24\pi\times10^6}{3\times10^8} = \omega\sqrt{\mu_0\epsilon_0}. \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.08\pi} = 25\text{ m.}$$

$$\vec{E}_0 = 5(-\sqrt{3}\,\vec{i}_x + \vec{i}_y)$$

$$\vec{\beta}\cdot\vec{E}_0 = (\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)\cdot(-\sqrt{3}\,\vec{i}_x + \vec{i}_y) = -3 + 3 + 0 = 0$$

$\therefore \vec{E}_0$ is $\perp$ to $\vec{\beta}$.

$$\vec{H} = \frac{1}{\omega\mu}\vec{\beta}\times\vec{E}$$

$$\vec{H}_0 = \frac{1}{\omega\mu}\vec{\beta}\times\vec{E}_0 = \frac{0.02\pi\times5}{24\pi\times10^6\times4\pi\times10^{-7}}(\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)\times(-\sqrt{3}\,\vec{i}_x + \vec{i}_y)$$

$$= \frac{1}{96\pi}(\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)\times(-\sqrt{3}\,\vec{i}_x + \vec{i}_y)$$

$$= \frac{1}{96\pi}\begin{vmatrix} \vec{i}_x & \vec{i}_y & \vec{i}_z \\ \sqrt{3} & 3 & 2 \\ -\sqrt{3} & 1 & 0 \end{vmatrix} = \frac{1}{96\pi}\left(-2\vec{i}_x - 2\sqrt{3}\,\vec{i}_y + 4\sqrt{3}\,\vec{i}_z\right)$$

$$= \frac{1}{48\pi}\left(-\vec{i}_x - \sqrt{3}\,\vec{i}_y + 2\sqrt{3}\,\vec{i}_z\right)$$

$$\therefore \vec{H} = \frac{1}{48\pi}\left(-\vec{i}_x - \sqrt{3}\,\vec{i}_y + 2\sqrt{3}\,\vec{i}_z\right)\cos\left[24\pi\times10^6 t - 0.02\pi(\sqrt{3}x + 3y + 2z)\right].$$

<u>Verification:</u>

$$\vec{\beta}\cdot\vec{H}_0 = (\sqrt{3}\,\vec{i}_x + 3\vec{i}_y + 2\vec{i}_z)\cdot(-\vec{i}_x - \sqrt{3}\,\vec{i}_y + 2\sqrt{3}\,\vec{i}_z)$$
$$= (-\sqrt{3} - 3\sqrt{3} + 4\sqrt{3}) = 0$$

$$\vec{E}_0\cdot\vec{H}_0 = (-\sqrt{3}\,\vec{i}_x + \vec{i}_y)\cdot(-\vec{i}_x - \sqrt{3}\,\vec{i}_y + 2\sqrt{3}\,\vec{i}_z)$$
$$= \sqrt{3} - \sqrt{3} = 0$$

We can proceed further and find a few other quantities as follows:

$$\beta_x = 0.02\pi\times\sqrt{3} = 0.02\sqrt{3}\,\pi$$

$$\beta_y = 0.02\pi\times 3 = 0.06\,\pi$$

$$\beta_z = 0.02\pi\times 2 = 0.04\,\pi$$

We define apparent wavelengths and apparent phase velocities along the coordinate axes as

$$\lambda_x = \frac{2\pi}{\beta_x} = \frac{2\pi}{0.02\sqrt{3}\pi} = 57.7\text{ m}, \quad v_{px} = \frac{\omega}{\beta_x} = \frac{24\pi\times10^6}{0.02\sqrt{3}\pi} = 6.928\times10^8\text{ m/sec}$$

$$\lambda_y = \frac{2\pi}{\beta_y} = \frac{2\pi}{0.06\pi} = 33.3\text{ m}, \quad v_{py} = \frac{\omega}{\beta_y} = \frac{24\pi\times10^6}{0.06\pi} = 4\times10^8\text{ m/sec}$$

$$\lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{0.04\pi} = 50\text{ m}, \quad v_{pz} = \frac{\omega}{\beta_z} = \frac{24\pi\times10^6}{0.04\pi} = 6\times10^8\text{ m/sec}.$$

Note that

$$\frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} + \frac{1}{\lambda_z^2} = \frac{1}{\lambda^2}, \qquad \frac{1}{v_{px}^2} + \frac{1}{v_{py}^2} + \frac{1}{v_{pz}^2} = \frac{1}{v_p^2}.$$

TRANSVERSE ELECTRIC WAVES:

Let us now consider the superposition of two uniform waves having electric fields

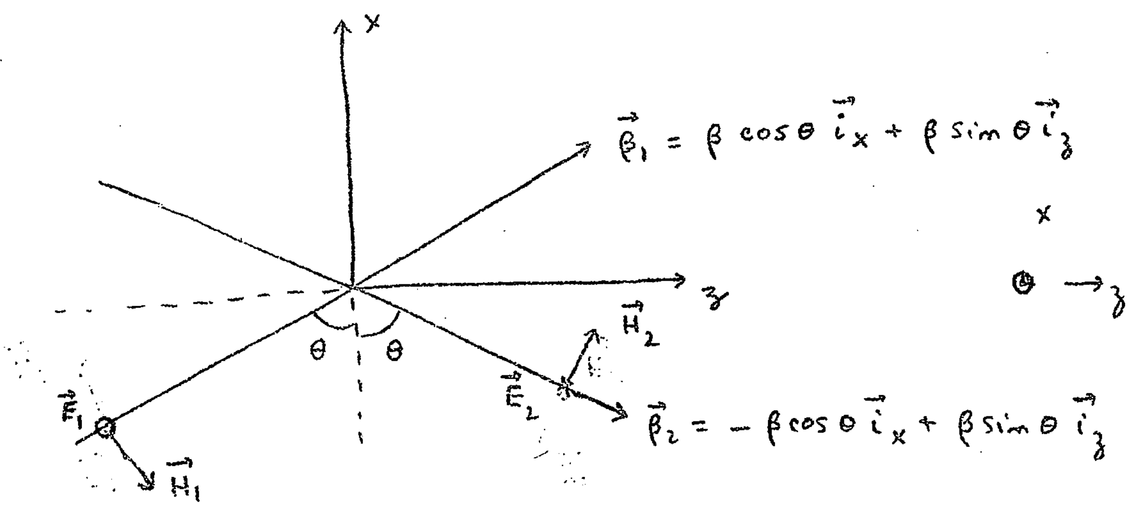
$$\vec{E}_1 = E_0 \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta)\ \vec{i}_y$$

$$\vec{E}_2 = -E_0 \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)\ \vec{i}_y$$

The corresponding magnetic fields are given by

$$\vec{H}_1 = \frac{E_0}{\eta}(-\sin\theta\,\vec{i}_x + \cos\theta\,\vec{i}_z)\cdot\cos(\omega t - \beta x\cos\theta - \beta z \sin\theta)$$

$$\vec{H}_2 = \frac{E_0}{\eta}(\sin\theta\,\vec{i}_x + \cos\theta\,\vec{i}_z)\cdot\cos(\omega t + \beta x\cos\theta - \beta z \sin\theta)$$



The electric field of the superposition of the two waves is given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = [E_0 \cos(\omega t - \beta z\sin\theta - \beta x\cos\theta) - E_0\cos(\omega t - \beta z \sin\theta + \beta z\cos\theta)]\,\vec{i}_y$$

$$= 2E_0 \underbrace{\sin(\beta x\cos\theta)}_{\substack{\text{standing wave}\\ \text{character in the}\\ x\text{ direction}}}\ \underbrace{\sin(\omega t - \beta z\sin\theta)}_{\substack{\text{Traveling wave}\\ \text{character in the}\\ z\text{ direction}}}\ \vec{i}_y.$$

The corresponding magnetic field is given by

$$\vec{H} = \vec{H}_1 + \vec{H}_2 = \left\{ \frac{-E_0 \sin\theta}{\eta} \left[\cos(\omega t - \beta z \sin\theta - \beta x \cos\theta) - \cos(\omega t - \beta z \sin\theta + \beta x \cos\theta) \right] \right\} \vec{i}_x$$

$$+ \left\{ \frac{E_0 \cos\theta}{\eta} \left[\cos(\omega t - \beta z \sin\theta - \beta x \cos\theta) + \cos(\omega t - \beta z \sin\theta + \beta x \cos\theta) \right] \right\} \vec{i}_y$$

$$= -\frac{2 E_0 \sin\theta}{\eta} \sin(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \sin\theta)\ \vec{i}_x$$

$$+ \frac{2 E_0 \cos\theta}{\eta} \underbrace{\cos(\beta x \cos\theta)}_{\text{Standing wave}} \cdot \underbrace{\cos(\omega t - \beta z \sin\theta)}_{\text{Traveling wave}}\ \vec{i}_z$$

Thus we have a standing wave in the x direction moving bodily in the z direction. Since the time average power flow is then entirely in the z direction and since the electric field is entirely transverse to the z direction whereas ~~as~~ the magnetic field is not, these waves are known as transverse electric or TE waves.

HOMEWORK PROBLEM (Due 11/20/74)

Given

$$\vec{E} = (\vec{i}_x - 2\vec{i}_y - \sqrt{3}\,\vec{i}_z) \cos\left[15\pi \times 10^6 t - 0.05\pi(\sqrt{3}x + 3)\right] \quad \text{V/m}$$

$$\vec{H} = \frac{1}{60\pi}(\vec{i}_x + 2\vec{i}_y - \sqrt{3}\,\vec{i}_z) \cos\left[15\pi \times 10^6 t - 0.05\pi(\sqrt{3}x + 3)\right] \quad \text{A/m}$$

a) Perform all the necessary tests to determine if these fields represent a uniform plane wave.

b) Find the permittivity and the permeability of the medium.

TRANSVERSE ELECTRIC WAVES:

In the previous lecture, we introduced uniform plane wave propagation in an arbitrary direction. Let us now consider the superposition of two uniform plane waves having electric fields

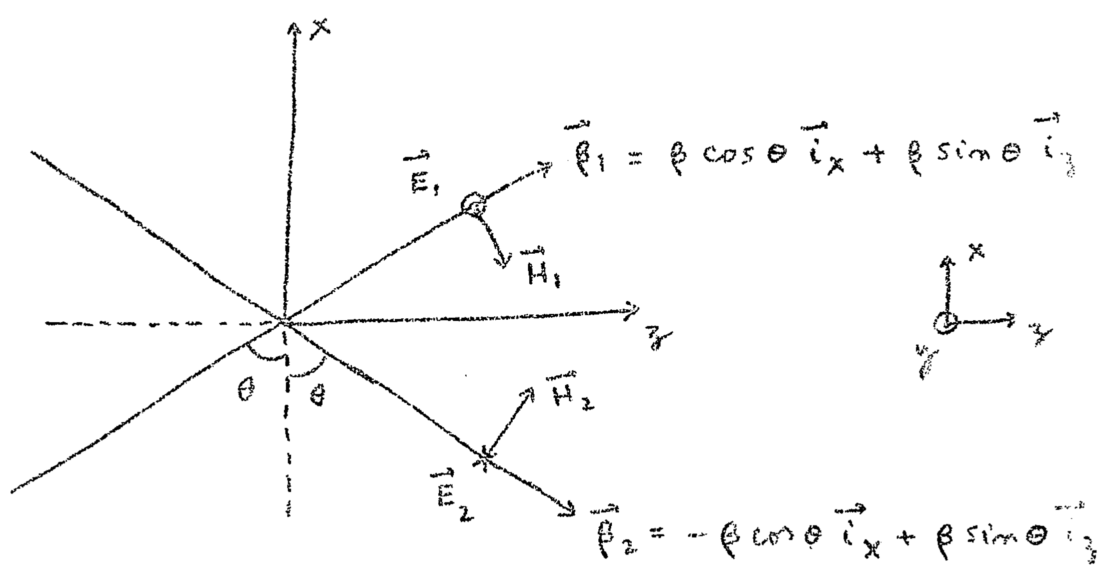
$$\vec{E}_1 = E_0 \cos(\omega t - \beta x \cos\theta - \beta z \sin\theta)\ \vec{i}_y$$

$$\vec{E}_2 = -E_0 \cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)\ \vec{i}_y$$

The corresponding magnetic fields are given by

$$\vec{H}_1 = \frac{E_0}{\eta}(-\sin\theta\ \vec{i}_x + \cos\theta\ \vec{i}_z)\cdot\cos(\omega t - \beta x \cos\theta - \beta z \sin\theta)$$

$$\vec{H}_2 = \frac{E_0}{\eta}(\sin\theta\ \vec{i}_x + \cos\theta\ \vec{i}_z)\cdot\cos(\omega t + \beta x \cos\theta - \beta z \sin\theta)$$



The electric and magnetic fields of the superposition of the two waves are given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \left[E_0 \cos(\omega t - \beta z \sin\theta - \beta x \cos\theta) - E_0 \cos(\omega t - \beta z \sin\theta + \beta x \cos\theta)\right]\vec{i}_y$$

$$= 2E_0 \sin(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \sin\theta)\,\vec{i}_y$$

$$\vec{H} = \vec{H}_1 + \vec{H}_2 = -\frac{E_0 \sin\theta}{\eta}\left[\cos(\omega t - \beta z \sin\theta - \beta x \cos\theta) - \cos(\omega t - \beta z \sin\theta + \beta x \cos\theta)\right]\vec{i}_x$$

$$+ \frac{E_0 \cos\theta}{\eta}\left[\cos(\omega t - \beta z \sin\theta - \beta x \cos\theta) + \cos(\omega t - \beta z \sin\theta + \beta x \cos\theta)\right]\vec{i}_z$$

$$= -\frac{2E_0 \sin\theta}{\eta} \sin(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \sin\theta)\,\vec{i}_x$$

$$+ \frac{2E_0 \cos\theta}{\eta} \underbrace{\cos(\beta x \cos\theta)}_{\substack{\text{standing wave} \\ \text{character in the} \\ x \text{ direction}}} \cdot \underbrace{\cos(\omega t - \beta z \sin\theta)}_{\substack{\text{Traveling wave} \\ \text{character in the} \\ z \text{ direction}}}\,\vec{i}_z$$

The fields have standing wave character in the x direction and traveling wave character in the z direction. Thus we have standing waves in the x direction moving bodily in the z direction. Since the time average power flow is then entirely in the z direction and since the electric field is entirely transverse to the z direction whereas the magnetic field is not, these waves are known as <u>transverse electric</u> or TE waves.

PARALLEL PLATE WAVEGUIDE:

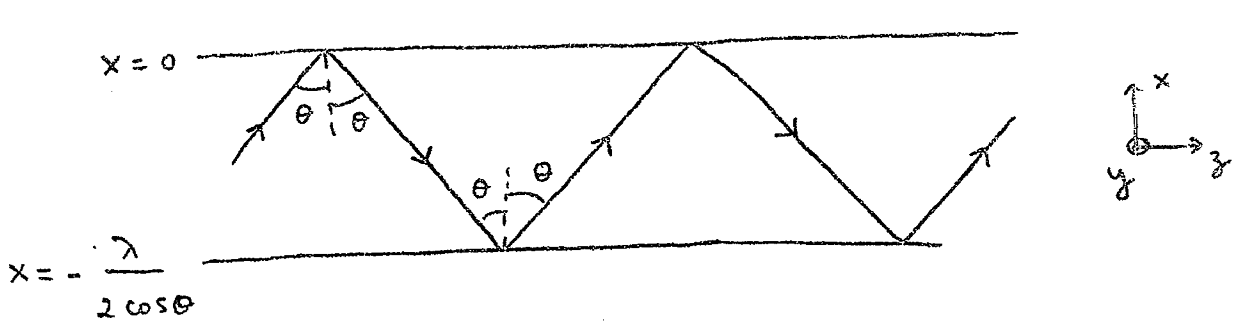
From the solutions for the fields, we note that the electric field of the superposition of the two waves is zero for

$$\sin(\beta x \cos\theta) = 0$$

or, $$\beta x \cos\theta = \pm m\pi \quad , \quad m = 0, 1, 2, 3, \ldots$$

or, $$x = \frac{\pm m\pi}{\beta\cos\theta} = \pm \frac{m\lambda}{2\cos\theta} \qquad \beta = \frac{2\pi}{\lambda}$$

Thus if we place perfectly conducting sheets in these planes, the waves will propagate undisturbed, i.e., as though the sheets were not there since the boundary condition that the tangential electric field be zero on the surface of the perfect conductor is satisfied. Note that the second boundary condition that the normal magnetic field be zero on the surface of the perfect conductor is also satisfied since H_x is also zero in the same planes. If we consider any two adjacent sheets, the situation is actually one of waves bouncing obliquely between the two sheets as shown below,

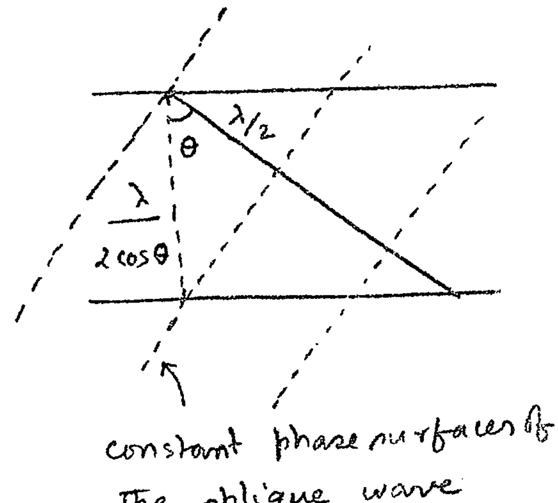


thereby guiding the wave and hence the energy in the z direction.

We thus have a parallel plate waveguide.

We note that $\frac{\lambda}{\cos\theta}$ is simply the apparent wavelength of the obliquely bouncing wave in the x direction, i.e., normal to the plates.

Thus the fields have one-half apparent wavelength in the x direction.



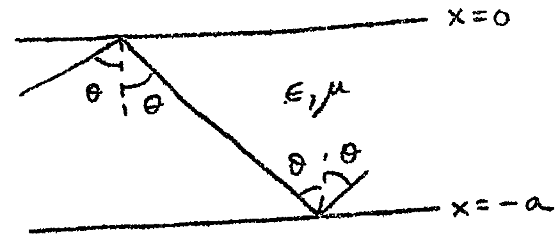
If we place the perfectly conducting sheets in the planes

$$x = 0 \quad \text{and} \quad x = -\frac{m\lambda}{2\cos\theta},$$

then the fields will have m number of apparent half wavelengths in the x-direction. Since there are no variations in the y direction, the fields are said to correspond to $TE_{m,0}$ modes. The subscript m refers to the x direction, denoting m number half apparent wavelengths. The subscript 0 refers to the y direction, denoting zero number of half apparend wavelengths in that direction.

Let us now consider to parallel perfectly conducting plates situated in the planes $x = 0$ and $x = -a$, i.e., having a fixed spacing a between them. Then for $TE_{m,0}$ modes propagating between the two plates,

$$a = \frac{m\lambda}{2\cos\theta}$$



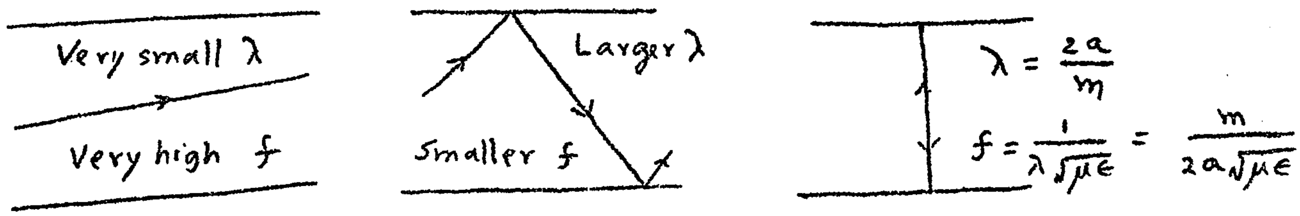
or $$\cos\theta = \frac{m\lambda}{2a}$$

Thus, waves of different wavelengths (or frequencies) bounce obliquely between the plates at different angles.

For very small wavelengths (very high frequencies), $\frac{m\lambda}{2a}$ is very small, $\cos\theta \approx 0$, $\theta \approx 90^\circ$, the waves simply slide between the plates as in the case of an ordinary transmission line.

As λ increases (f decreases), $\frac{m\lambda}{2a}$ increases, θ decreases, the waves bounce more and more obliquely, until λ becomes $\frac{2a}{m}$ when $\cos\theta = 1$, $\theta = 0^\circ$ and the waves simply bounce up and down normally to the plates without any feeling of being guided parallel to the plates.

For $\lambda > \frac{2a}{m}$, $\frac{m\lambda}{2a} > 1$, $\cos\theta > 1$, θ has no real solution.



Thus we have a situation in which propagation is cut off for $\lambda > \frac{2a}{m}$ or $f < \frac{m}{2a\sqrt{\mu\epsilon}}$. These are known as the <u>cut off wavelength</u> λ_c and the <u>cut off frequency</u> f_c.

$$\boxed{\lambda_c = \frac{2a}{m}, \quad f_c = \frac{m}{2a\sqrt{\mu\epsilon}}}$$

Substituting λ_c for $\frac{2a}{m}$ in the expression for $\cos\theta$, we have

$$\cos\theta = \frac{\lambda}{\lambda_c} = \frac{f_c}{f}, \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2} = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Noting then that

$$\beta\cos\theta = \frac{2\pi}{\lambda}\cdot\frac{\lambda}{\lambda_c} = \frac{2\pi}{\lambda_c} = \frac{m\pi}{a}$$

$$\beta\sin\theta = \frac{2\pi}{\lambda}\cdot\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}$$

we see that the phase constant along the z direction, i.e., $\beta\sin\theta$ is real for $\lambda < \lambda_c$ and imaginary for $\lambda > \lambda_c$. This, once again, explains the cut off phenomenon. We now define the <u>guide wavelength</u>, λ_g, to the wavelength in the z direction.

$$\lambda_g = \frac{2\pi}{\beta\sin\theta} = \lambda\Big/\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2} = \lambda\Big/\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

This is simply the apparent wavelength in the z-direction.

EXAMPLE: $a = 5$ cm, $f = 10{,}000$ MHz. We wish to determine all propagating $TE_{m,0}$ modes, assuming free space between the plates.

$$\lambda_c = \frac{2a}{m} = \frac{10}{m} \text{ cm}$$

$f = 10{,}000$ MHz corresponds to $\lambda = 3$ cm in free space

Hence the propagating $TE_{m,0}$ modes are

$TE_{1,0}$ ($\lambda_c = 10$ cm), $TE_{2,0}$ ($\lambda_c = 5$ cm) and $TE_{3,0}$ ($\lambda_c = \frac{10}{3}$ cm).

For each propagating mode, we can find

$$f_c = \frac{m}{2a\sqrt{\mu_0\epsilon_0}}, \quad \theta = \cos^{-1}\frac{\lambda}{\lambda_c} \text{ and } \lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}.$$

These are given below.

Mode	$TE_{1,0}$	$TE_{2,0}$	$TE_{3,0}$
f_c, MHz	3000	6000	9000
θ, deg.	72.55	53.13	25.15
λ_g, cm	3.145	3.75	6.883

HOMEWORK PROBLEM (Due 11/22/74)

The dimension a of a parallel-plate waveguide filled with a dielectric having $\epsilon = 4\epsilon_0$ and $\mu = \mu_0$ is 4 cm. Determine the propagating $TE_{m,0}$ modes for a wave of frequency 6000 MHz. For each propagating mode, find f_c, θ, and λ_g.

$TE_{m,0}$ FIELDS IN A PARALLEL-PLATE GUIDE :

We recall from the previous lecture that we considered the superposition of two obliquely propagating uniform plane waves giving rise to standing wave character in the x direction and traveling wave character in the z direction. From page 30-2, the field expressions for the superposition of the two waves are

$$\vec{E} = 2E_0 \sin(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \quad \theta)\ \vec{i}_y$$

$$\vec{H} = -\frac{2E_0 \sin\theta}{\eta} \sin(\beta x \cos\theta) \cdot \sin(\omega t - \beta z \sin\theta)\ \vec{i}_x$$

$$+ \frac{2E_0 \cos\theta}{\eta} \cos(\beta x \cos\theta) \cdot \cos(\omega t - \beta z \sin\theta)\ \vec{i}_z$$

We then placed perfect conductors in the planes $x = 0$ and $x = -a$ and obtained the parallel-plate waveguide. Since the tangential electric field (E_y) and the normal magnetic field (H_x) must be zero on the perfect conductors, we found that

$$\sin(\beta a \cos\theta) = 0$$

$$\beta a \cos\theta = m\pi \quad \text{or} \quad \beta\cos\theta = \frac{m\pi}{a}.$$

We also defined the guide wavelength λ_g, which is simply the apparent wavelength of the oblique wave in the z-direction, as

$$\lambda_g = \frac{2\pi}{\beta \sin\theta}$$

Thus, substituting $\beta\cos\theta = \frac{m\pi}{a}$ and $\beta\sin\theta = \frac{2\pi}{\lambda_g}$ in the field expressions, we get

$$\vec{E} = 2E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_y$$

$$\vec{H} = -\frac{2E_0}{\eta}\frac{\lambda}{\lambda_g} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_x$$

$$+ \frac{2E_0}{\eta}\frac{\lambda}{\lambda_c} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_z$$

where we have also used $\cos\theta = \frac{\lambda}{\lambda_c}$ and $\sin\theta = \frac{2\pi}{\beta\lambda_g} = \frac{\lambda}{\lambda_g}$.

These expressions for the fields in the parallel plate waveguide are independent of the angle θ, i.e., they do not contain θ. They clearly indicate standing wave character in the x direction having m half-sinusoidal variations. Also, the Poynting vector is given by

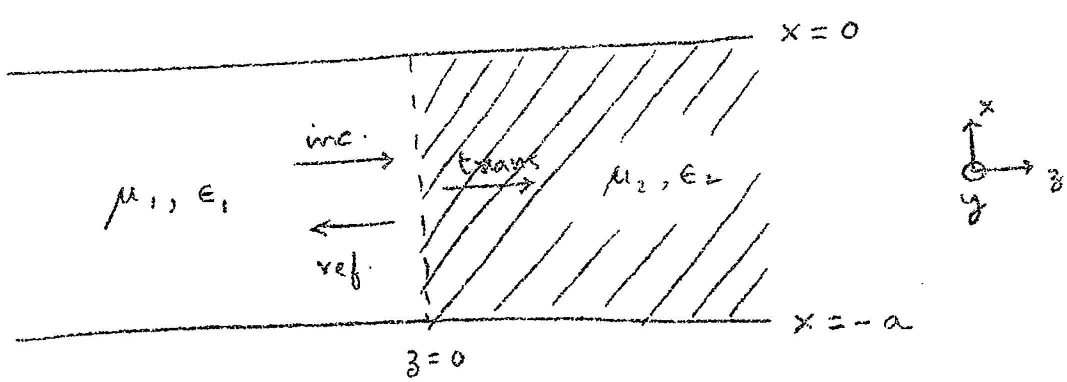
$$\vec{P} = \vec{E} \times \vec{H} = E_y \vec{i}_y \times (H_x \vec{i}_x + H_z \vec{i}_z) = -E_y H_x \vec{i}_z + E_y H_z \vec{i}_x$$

$$= \frac{4E_0^2}{\eta}\frac{\lambda}{\lambda_g} \sin^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_z$$

$$+ \frac{E_0^2}{\eta}\frac{\lambda}{\lambda_c} \sin\left(\frac{2m\pi x}{a}\right) \sin 2\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_x$$

Thus the time average power flow is entirely in the z-direction, i.e., parallel to the plates. Since $\vec{E}$ is entirely transverse to this direction and since it has m half-sinusoidal variations in the x direction and zero half-sinusoidal variations in the y direction, the fields are said to correspond to $TE_{m,0}$ modes as discussed before.

PARALLEL-PLATE WAVEGUIDE DISCONTINUITY



Let us now consider reflection and transmission at a discontinuity in a parallel-plate waveguide as shown above. If a $TE_{m,0}$ wave is incident on the junction from section 1, then it will set up a reflected wave and a transmitted wave. The fields in these waves must be such that the boundary conditions are satisfied at the dielectric discontinuity. Denoting the incident, reflected, and transmitted wave fields by subscripts i, r, and t, respectively, we have at the junction $z=0$,

$$E_{yi} + E_{yr} = E_{yt} \qquad \text{tangential electric field intensity} \quad -(1)$$

$$H_{xi} + H_{xr} = H_{xt} \qquad \text{tangential magnetic field intensity}$$

or

$$\frac{E_{yi}}{-\dfrac{E_{yi}}{H_{xi}}} + \frac{E_{yr}}{-\dfrac{E_{yr}}{H_{xr}}} = \frac{E_{yt}}{-\dfrac{E_{yt}}{H_{xt}}}$$

We now define the guide impedance, η_g, as

$$\eta_g = \frac{E_{yi}}{-H_{xi}} = \frac{\eta \lambda_g}{\lambda} = \eta \Big/ \sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2} = \eta \Big/ \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

The guide impedance is analogous to the characteristic impedance in the case of a transmission line. The second boundary condition can now be written as

$$\frac{E_{yi}}{\eta_{g1}} - \frac{E_{yr}}{\eta_{g1}} = \frac{E_{yt}}{\eta_{g2}} \qquad -(2)$$

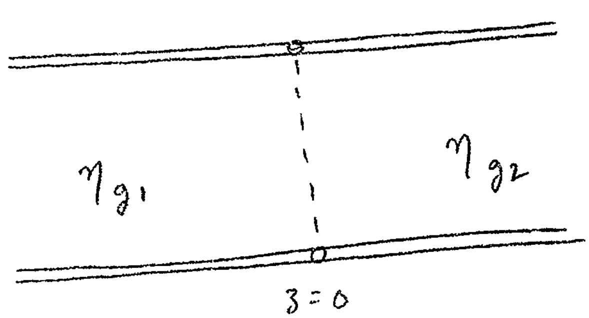
Solving (1) and (2), we get

$$E_{yi}\left(1-\frac{\eta_{g2}}{\eta_{g1}}\right) + E_{yr}\left(1+\frac{\eta_{g2}}{\eta_{g1}}\right) = 0$$

$$\Gamma = \frac{E_{yr}}{E_{yi}} = \frac{\eta_{g2}-\eta_{g1}}{\eta_{g2}+\eta_{g1}}$$

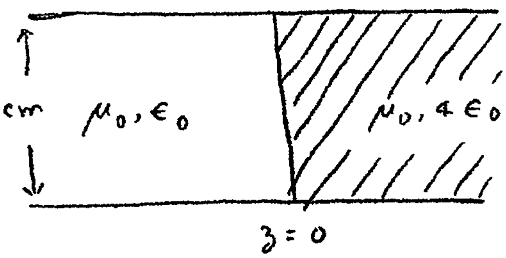
$$\tau = \frac{E_{yt}}{E_{yi}} = \frac{E_{yi}+E_{yr}}{E_{yi}} = 1+\Gamma.$$

The situation is analogous to a transmission-line discontinuity as shown below.



EXAMPLE:

$a = 3$ cm



$TE_{1,0}$ waves of $f = 6000$ MHz are incident from the free space side. We wish to find the reflection coefficient Γ.

For $TE_{1,0}$ mode, $\lambda_c = 2a = 6$ cm (Note that λ_c does not depend upon the dielectric)

For $f = 6000$ MHz

wavelength on the free space side = 5 cm

wavelength in the dielectric $= \frac{5}{\sqrt{4}} = 2.5$ cm.

Since $\lambda < \lambda_c$ in both sections, $TE_{1,0}$ mode propagates in both sections. Now

$$\eta_{g1} = \frac{\eta_1}{\sqrt{1-(\lambda_1/\lambda_c)^2}} = \frac{120\pi}{\sqrt{1-(5/6)^2}} = 682\ \Omega$$

$$\eta_{g2} = \frac{\eta_2}{\sqrt{1-(\lambda_2/\lambda_c)^2}} = \frac{120\pi/\sqrt{4}}{\sqrt{1-(2.5/6)^2}} = \frac{60\pi}{\sqrt{1-(2.5/6)^2}} = 207.35\ \Omega$$

$$\therefore \Gamma = \frac{\eta_{g2}-\eta_{g1}}{\eta_{g2}+\eta_{g1}} = \frac{207.35-682}{207.35+682} = -0.5337.$$

We can proceed further and find the fraction of the incident power reflected. This is simply $\Gamma^2 = 0.5337^2 = 0.285$.

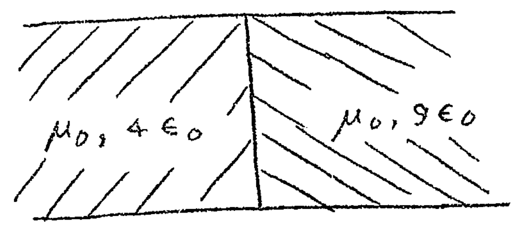
The fraction of the incident power transmitted is then equal to $1 - 0.285 = 0.715$.

HOMEWORK PROBLEM (Due 11/25/74)

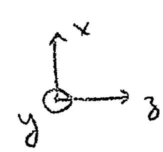
Repeat the example problem on the previous page for the following data:

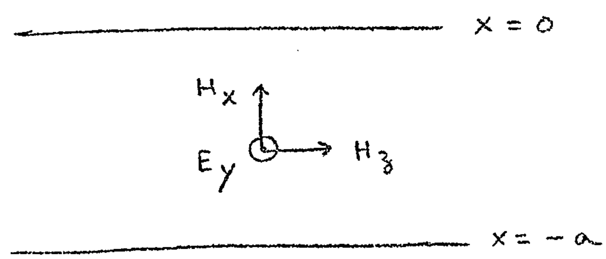
$a = 4$ cm

$f = 2500$ MHz

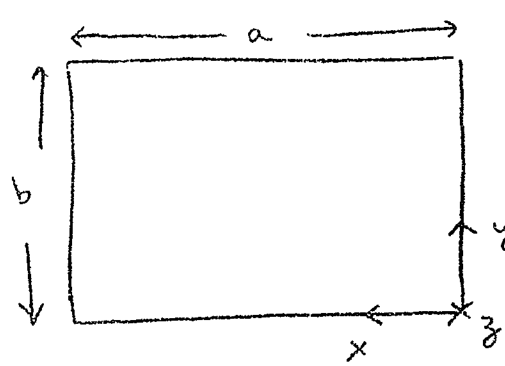


EXTENSION TO RECTANGULAR WAVEGUIDE





Since $\vec{E}$ has only a y component, the fields will not be disturbed if we place in the y = constant planes, i.e., normal to the electric field. We then have a rectangular waveguide.



RECTANGULAR CAVITY RESONATOR

In the previous lecture, we studied $TE_{m,0}$ modes in a parallel-plate waveguide. We found that the solutions for the fields are given by

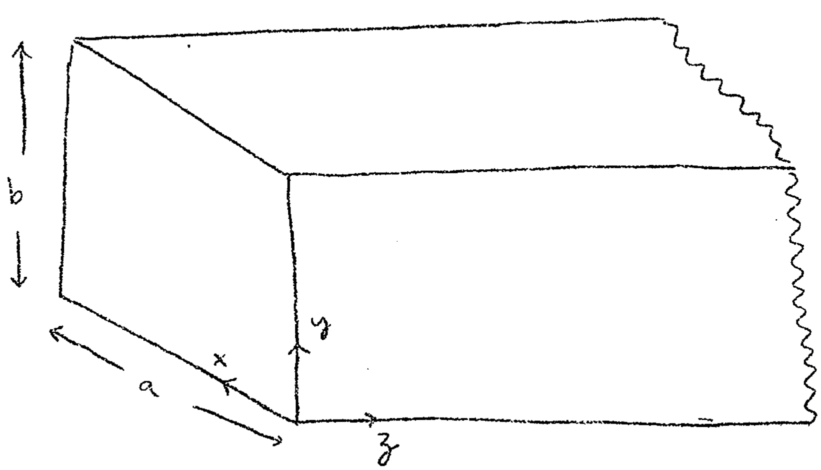
$$\vec{E} = 2E_0 \sin\frac{m\pi x}{a} \sin\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_y$$

$$\vec{H} = -\frac{2E_0\lambda}{\eta\lambda_g} \sin\frac{m\pi x}{a} \sin\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_x$$

$$+ \frac{2E_0}{\eta}\frac{\lambda}{\lambda_c} \cos\frac{m\pi x}{a} \cos\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_z$$

for a guide with the plates in the planes $x=0$ and $x=a$.

We then learned that since $\vec{E}$ has only a y component, the fields will remain undisturbed if we place two more perfect conductors in the y = constant planes, i.e., normal to the electric field, thereby leading to $TE_{m,0}$ modes in a rectangular waveguide.



Let us now consider $TE_{m,0}$ waves of equal magnitude propagating in the positive z and negative z directions, i.e., having electric fields,

$$\vec{E}_1 = 2E_0 \sin\frac{m\pi x}{a} \sin\left(\omega t - \frac{2\pi}{\lambda_g} z\right) \vec{i}_y$$

$$\vec{E}_2 = -2E_0 \sin\frac{m\pi x}{a} \sin\left(\omega t + \frac{2\pi}{\lambda_g} z\right) \vec{i}_y$$

The electric field of the superposition of these two waves is given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = -2E_0 \sin\frac{m\pi x}{a} \sin\frac{2\pi}{\lambda_g} z \cos\omega t \; \vec{i}_y$$

which corresponds to standing waves in the z direction also, since there exist nulls in the field for

$$\sin\frac{2\pi}{\lambda_g} z = 0 \quad \text{or} \quad \frac{2\pi}{\lambda_g} z = l\pi \;, \quad l = 0, 1, 2, 3, \ldots$$

$$\text{or} \quad z = \frac{l}{2}\lambda_g \;, \quad l = 0, 1, 2, 3, \ldots$$

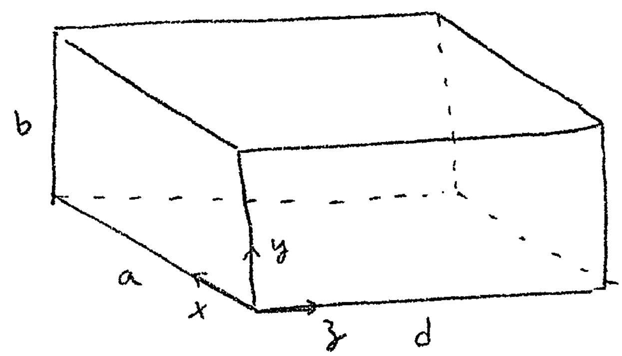
Placing of perfect conductors in these planes will not disturb the fields since the boundary condition of zero tangential electric field is satisfied.

Conversely, if we place two perfect conductors in two constant z planes separated by a distance d, then in order for the above condition to be satisfied,

$$d \text{ must be } = \frac{l\lambda_g}{2}.$$

We then have a rectangular cavity resonator having dimensions a, b, and d in the x, y, and z directions, respectively, and oscillating in the $TE_{m,0,l}$ mode, i.e., with m half sine variations in the x direction, zero half sine variations in the y direction and l half sine variations in the z directions.

This is the counterpart of the ordinary resonant circuit at microwave frequencies. The frequency of oscillation is governed by the condition



$$d = \frac{l\lambda_g}{2} = \frac{l}{2}\frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$$

or,
$$1 - \frac{\lambda^2}{\lambda_c^2} = \left(\frac{l\lambda}{2d}\right)^2$$

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_c^2} + \left(\frac{l}{2d}\right)^2 = \left(\frac{m}{2a}\right)^2 + \left(\frac{l}{2d}\right)^2$$

$$\frac{1}{\lambda} = \sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{l}{2d}\right)^2}$$

Thus

$$f_{osc} = \frac{v_p}{\lambda} = \frac{1}{\sqrt{\mu\epsilon}} \cdot \sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{l}{2d}\right)^2}$$

EXAMPLE: $a = 4$ cm, $b = 2$ cm, $d = 4$ cm

$\mu = \mu_0$, $\epsilon = \epsilon_0$

Frequency of oscillation for the TE_{101} mode

$$= \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{1}{8 \times 10^{-2}}\right)^2 + \left(\frac{1}{8 \times 10^{-2}}\right)^2}$$

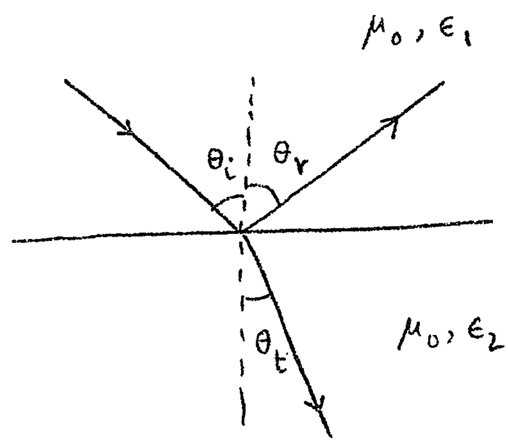
$$= 3 \times 10^8 \times \frac{\sqrt{2}}{8 \times 10^{-2}} = 3750 \times \sqrt{2} \text{ MHz}$$

$$= 5.303 \text{ GHz}.$$

OPTICAL WAVEGUIDES:

Let us consider an obliquely incident uniform plane wave at a plane boundary between two different dielectric media. Let θ_i, θ_r and θ_t be the angle of incidence, angle of reflection and angle of refraction, respectively, as shown in the figure.

Then in order for the incident, reflected and refracted waves to be in step at the boundary between the dielectrics, their apparent phase velocities of propagation parallel to the boundary must be equal.

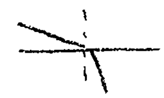


Thus, $\dfrac{v_{p1}}{\sin\theta_i} = \dfrac{v_{p0}}{\sin\theta_r} = \dfrac{v_{p2}}{\sin\theta_t}$

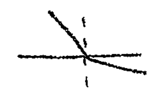
or $\sin\theta_r = \sin\theta_i$, $\boxed{\theta_r = \theta_i}$

$$\boxed{\sin\theta_t = \frac{v_{p2}}{v_{p1}}\sin\theta_i = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i}$$ Snell's law

For $\epsilon_2 > \epsilon_1$, $\sin\theta_t < \sin\theta_i$, $\theta_t < \theta_i$



For $\epsilon_2 < \epsilon_1$, $\sin\theta_t > \sin\theta_i$, $\theta_t > \theta_i$

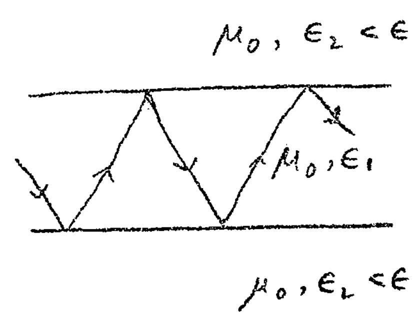


There exists a value of θ_i for which $\theta_t = 90°$.

This is given by $\sin\theta_i = \sqrt{\dfrac{\epsilon_2}{\epsilon_1}}$ or $\theta_i = \sin^{-1}\dfrac{\epsilon_2}{\epsilon_1}$.

For $\theta_i > \sin^{-1}\dfrac{\epsilon_2}{\epsilon_1}$, total internal reflection occurs.

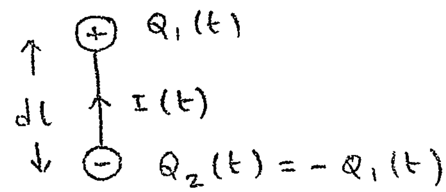
Hence, if we have a dielectric medium of permittivity ϵ_1, sandwitched between dielectric media of permittivity $\epsilon_2 < \epsilon_1$, then by lauching waves at angles of incidence greater than $\sin^{-1}\dfrac{\epsilon_2}{\epsilon_1}$, it is possible to have guided wave propagation. This is the principle of optical waveguides.



HERTZIAN DIPOLE

The Hertzian dipole is an elemental antenna consisting of an infinitesimally long piece of wire carrying an alternating current. To maintain the current flow, we can assume that the wire is terminated by point charges at the two ends.

Thus, if $I(t) = I_0 \cos \omega t$, then



Hertzian dipole

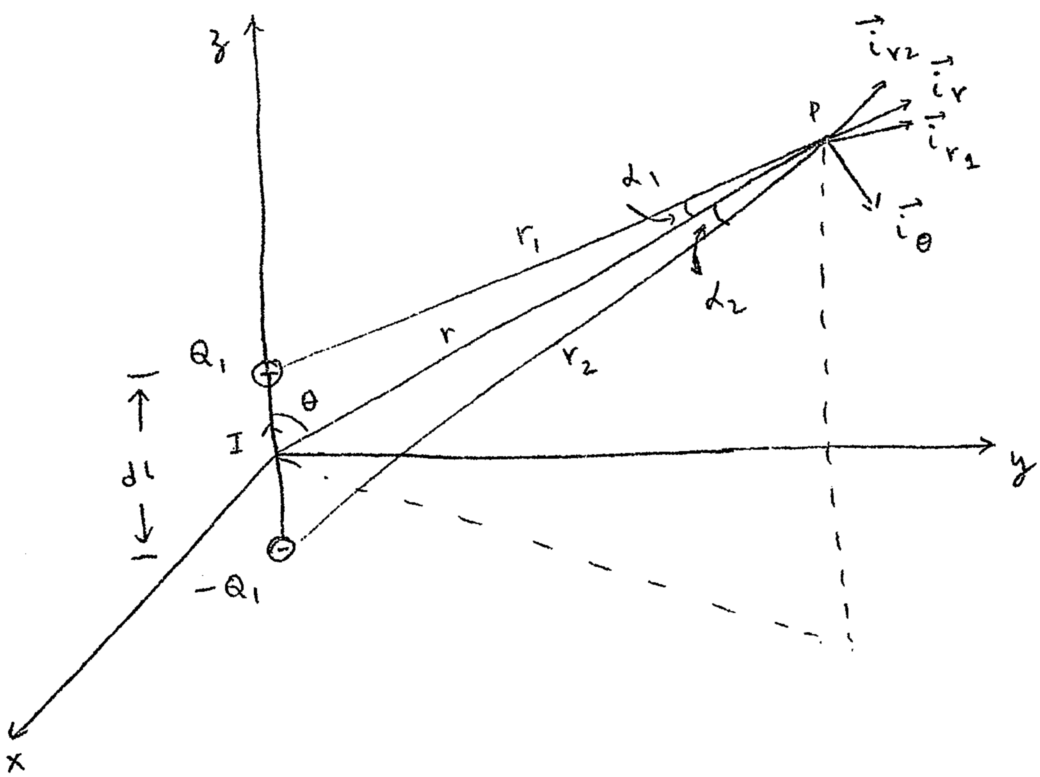
$$\frac{dQ_1}{dt} = I(t) = I_0 \cos \omega t$$

$$\frac{dQ_2}{dt} = -I(t) = I_0 \cos \omega t$$

$$Q_1(t) = \frac{I_0}{\omega} \sin \omega t$$

$$Q_2(t) = -\frac{I_0}{\omega} \sin \omega t = -Q_1(t)$$

Let us locate the Hertzian dipole at the origin, oriented along the z axis and find the electromagnetic field due to the dipole at a point $P(r, \theta, \phi)$. The approach we shall use here is to try an intuitive extension of static field considerations to the time varying case from our knowledge of wave propagation and then check if the solutions satisfy Maxwell's equations. If they do not, we then have to modify the solutions so that they do satisfy Maxwell's equations and at the same time are consistent with static field considerations.



From our knowledge of electric field due to a point charge, we can write the expression for the electric field at point P due to the arrangement of the two point charges to be

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 r_1^2}\vec{i}_{r_1} - \frac{Q_1}{4\pi\epsilon_0 r_2^2}\vec{i}_{r_2}$$

$$= \frac{Q_1}{4\pi\epsilon_0 r_1^2}(\cos\alpha_1\vec{i}_r + \sin\alpha_1\vec{i}_\theta)$$

$$- \frac{Q_1}{4\pi\epsilon_0 r_2^2}(\cos\alpha_2\vec{i}_r - \sin\alpha_2\vec{i}_\theta)$$

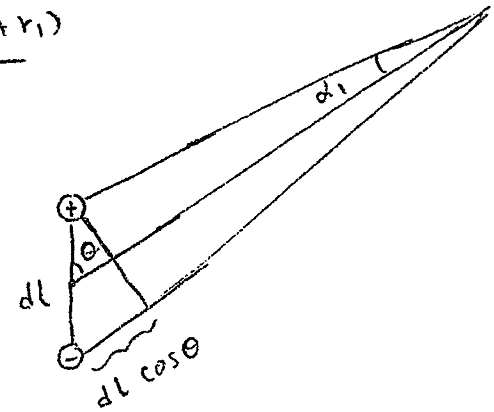
$$E_r = \frac{Q_1}{4\pi\epsilon_0}\left(\frac{\cos\alpha_1}{r_1^2} - \frac{\cos\alpha_2}{r_2^2}\right)$$

$$\approx \frac{Q_1}{4\pi\epsilon_0}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \qquad dl \ll r$$

$$= \frac{Q_1}{4\pi\epsilon_0}\,\frac{r_2^2 - r_1^2}{r_1^2 r_2^2} = \frac{Q_1}{4\pi\epsilon_0}\,\frac{(r_2-r_1)(r_2+r_1)}{r_1^2 r_2^2}$$

$$\approx \frac{Q_1\, dl\cos\theta}{4\pi\epsilon_0}\,\frac{2r}{r^4}$$

$$= \frac{Q_1 dl\, 2\cos\theta}{4\pi\epsilon_0 r^3}$$



$$E_\theta = \frac{Q_1}{4\pi\epsilon_0}\left(\frac{\sin\alpha_1}{r_1^2} + \frac{\sin\alpha_2}{r_2^2}\right)$$

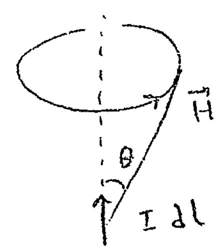
$$\approx \frac{Q_1}{4\pi\epsilon_0 r^2}\,2\sin\alpha_1 = \frac{Q_1}{4\pi\epsilon_0 r^2}\,2\,\frac{\frac{dl}{2}\sin\theta}{r_1}$$

$$\approx \frac{Q_1\, dl}{4\pi\epsilon_0 r^3}\sin\theta$$

Thus $\vec{E} = \dfrac{Q_1\, dl}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\vec{i}_r + \sin\theta\,\vec{i}_\theta\right)$

From our knowledge of Biot-Savart law, we obtain the magnetic field due to the current element as

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{I\, dl}{4\pi r^2}\sin\theta\,\vec{i}_\phi$$



The field expressions which we have obtained are based on static field considerations. When the charges and currents are varying with time as in the present case, the fields also vary with time but based on what we have studied thus far time varying electric and magnetic fields give rise to wave propagation and the effect of the time variations of the source quantities is felt at a point in space at a later value of time. This time lag is equal to the time it takes for the wave to propagate from the source point to the observation point.

Thus for

$$Q_1 = \frac{I_0}{\omega} \sin \omega t$$

$$I = I_0 \cos \omega t$$

we would intuitively expect the fields at point P to be given by

$$\vec{E} = \frac{I_0\, dl\, \sin(\omega t - \beta r)}{4\pi \epsilon_0 \omega r^3} \left(2\cos\theta\, \vec{i}_r + \sin\theta\, \vec{i}_\theta \right)$$

$$\vec{H} = \frac{I_0\, dl\, \cos(\omega t - \beta r)}{4\pi r^2} \sin\theta\, \vec{i}_\phi$$

where $\beta = \dfrac{\omega}{v_p}$ = phase constant

$v_p = \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$ = phase velocity

But there is one thing wrong with our intuitive expectation of the fields due to the Hertzian dipole! They do not satisfy Maxwell's curl equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (\vec{J} = 0 \text{ in free space})$$

For example, let us try the curl equation for $\vec{H}$. First we note that the expansion for the curl of a vector in spherical coordinates is

$$\vec{\nabla} \times \vec{A} = \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\vec{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\vec{i}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\vec{i}_\phi$$

Thus

$$\vec{\nabla} \times \vec{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{I_0\, dl \cos(\omega t - \beta r)}{4\pi r^2}\sin^2\theta\right]\vec{i}_r - \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{I_0\, dl \cos(\omega t - \beta r)}{4\pi r}\sin\theta\right]\vec{i}_\theta$$

$$= \frac{I_0\, dl \cos(\omega t - \beta r)}{4\pi r^3}\left(2\cos\theta\, \vec{i}_r + \sin\theta\, \vec{i}_\theta\right) - \frac{\beta I_0\, dl \sin(\omega t - \beta r)}{4\pi r^2}\sin\theta\, \vec{i}_\theta$$

$$= \epsilon_0 \frac{\partial \vec{E}}{\partial t} - \frac{\beta I_0\, dl \sin(\omega t - \beta r)}{4\pi r^2}\sin\theta\, \vec{i}_\theta$$

To resolve this discrepancy, we modify our solution for $\vec{E}$ to read as

$$\vec{E} = \frac{I_0 dl \sin(\omega t - \beta r)}{4\pi \epsilon_0 \omega r^3}(2\cos\theta \, \vec{i}_r + \sin\theta \, \vec{i}_\theta)$$

$$+ \frac{\beta I_0 dl \cos(\omega t - \beta r)}{4\pi \epsilon_0 \omega r^2} \sin\theta \, \vec{i}_\theta$$

Very close to the dipole, i.e., for small r, this additional term is small compared to the original expression. This is consistent with the fact that near to the dipole, propagation effects are negligible and hence the field should be predominantly the same as that obtained from intuitive extension of static field considerations.

We now check if this modified $\vec{E}$ and the original $\vec{H}$ satisfy the curl equation for $\vec{E}$. When we do this, we will find the curl equation is not satisfied and hence another modification is required for the field expression. After about three tries like this, we will finally obtain solutions for $\vec{E}$ and $\vec{H}$ which satisfy both curl equations. These solutions are given by

$$\boxed{\begin{aligned} \vec{E} &= \frac{2 I_0 dl \cos\theta}{4\pi\epsilon_0}\left[\frac{\sin(\omega t - \beta r)}{\omega r^3} + \frac{\cos(\omega t - \beta r)}{v_p r^2}\right]\vec{i}_r \\ &+ \frac{I_0 dl \sin\theta}{4\pi\epsilon_0}\left[\frac{\sin(\omega t - \beta r)}{\omega r^3} + \frac{\cos(\omega t - \beta r)}{v_p r^2} - \frac{\omega \sin(\omega t - \beta r)}{v_p^2 r}\right]\vec{i}_\theta \\ \vec{H} &= \frac{I_0 dl \sin\theta}{4\pi}\left[\frac{\cos(\omega t - \beta r)}{r^2} - \frac{\omega \sin(\omega t - \beta r)}{v_p r}\right]\vec{i}_\phi \end{aligned}}$$

These solutions for $\vec{E}$ and $\vec{H}$ due to the Hertzian dipole satisfy Maxwell's equations (see Homework Problem below) and also very close to the dipole, the dominant terms are those involving $\frac{1}{r^3}$, which have been obtained from static field considerations. Very far from the dipole, i.e., for large r, the dominant terms are those involving $\frac{1}{r}$. These terms are said to correspond to the radiation field of the antenna. The radiation field for the Hertzian dipole is

$$\vec{E} = -\frac{I_0\, dl\, \omega \sin\theta}{4\pi \epsilon_0 v_p^2 r} \sin(\omega t - \beta r)\, \vec{i}_\theta$$

$$\vec{H} = -\frac{I_0\, dl\, \omega \sin\theta}{4\pi v_p r} \sin(\omega t - \beta r)\, \vec{i}_\phi$$

We shall study these fields in the next lecture.

Homework Problem (due 12/4/74)

Verify that the solutions for $\vec{E}$ and $\vec{H}$ given on the previous page do indeed satisfy Maxwell's equations.

RADIATION CHARACTERISTICS OF A HERTZIAN DIPOLE:

In the previous lecture, we found the complete electromagnetic field due to a Hertzian dipole of length dl and current $I = I_0 \cos \omega t$, located at the origin. These fields are given by

$$\vec{E} = \frac{2 I_0\, dl \cos\theta}{4\pi\epsilon_0}\left[\frac{\sin(\omega t-\beta r)}{\omega r^3} + \frac{\cos(\omega t-\beta r)}{v_p r^2}\right]\vec{i}_r$$

$$+ \frac{I_0\, dl \sin\theta}{4\pi\epsilon_0}\left[\frac{\sin(\omega t-\beta r)}{\omega r^3} + \frac{\cos(\omega t-\beta r)}{v_p r^2} - \frac{\omega \sin(\omega t-\beta r)}{v_p^2 r}\right]\vec{i}_\theta$$

$$\vec{H} = \frac{I_0\, dl \sin\theta}{4\pi}\left[\frac{\cos(\omega t-\beta r)}{r^2} - \frac{\omega \sin(\omega t-\beta r)}{v_p r}\right]\vec{i}_\phi$$

Very far from the dipole, i.e., for large r, the predominant terms are the $\frac{1}{r}$ terms and the field expressions reduce to

$$\vec{E} = -\frac{I_0\, dl\, \omega \sin\theta}{4\pi\epsilon_0 v_p^2 r}\sin(\omega t-\beta r)\,\vec{i}_\theta = -\frac{I_0\, dl\, \beta\eta_0 \sin\theta}{4\pi r}\sin(\omega t-\beta r)\,\vec{i}_\theta$$

$$\vec{H} = -\frac{I_0\, dl\, \omega \sin\theta}{4\pi v_p r}\sin(\omega t-\beta r)\,\vec{i}_\phi = -\frac{I_0\, dl\, \beta \sin\theta}{4\pi r}\sin(\omega t-\beta r)\,\vec{i}_\phi$$

These fields are known as the "<u>radiation fields</u>" because they are the components which contribute to radiation of electromagnetic waves away from the dipole.

We note that the radiation fields are locally uniform plane waves, i.e. over any infinitesimal area at a given point (r, θ, ϕ), since $\vec{E}$, $\vec{H}$ and the direction of propagation (r direction) are perpendicular to each other and

$$\frac{|\vec{E}|}{|\vec{H}|} = \eta_0 .$$

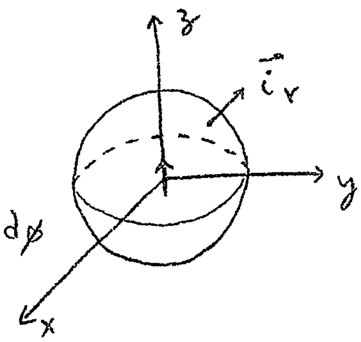
The Poynting vector due to the radiation fields is given by

$$\vec{P} = \vec{E} \times \vec{H} = -\frac{I_0 dl\, \beta \eta_0 \sin\theta}{4\pi r} \sin(\omega t - \beta r)\, \vec{i}_\theta \times \frac{-I_0 dl\, \beta \sin\theta}{4\pi r} \sin(\omega t - \beta r)\, \vec{i}_\phi$$

$$= \frac{I_0^2 (dl)^2 \beta^2 \eta_0 \sin^2\theta}{16\pi^2 r^2} \sin^2(\omega t - \beta r)\ \vec{i}_r$$

By evaluating the surface integral of the Poynting vector over the surface of radius r centered at the dipole, we obtain the power radiated by the antenna to be

$$P = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \vec{P} \cdot r^2 \sin\theta \, d\theta \, d\phi \ \vec{i}_r$$



$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{I_0^2 (dl)^2 \beta^2 \eta_0 \sin^3\theta}{16\pi^2} \sin^2(\omega t - \beta r)\, d\theta \, d\phi$$

$$= \frac{I_0^2 (dl)^2 \beta^2 \eta_0}{16\pi^2} \sin^2(\omega t - \beta r) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin^3\theta \, d\theta \, d\phi$$

$$= \frac{I_0^2 (dl)^2 \beta^2 \eta_0}{16\pi^2} \sin^2(\omega t - \beta r) \cdot 2\pi \cdot \frac{4}{3}$$

$$= \frac{\eta_0 I_0^2 (dl)^2 4\pi^2}{6\pi \lambda^2} \sin^2(\omega t - \beta r) = \frac{2\pi \eta_0 I_0^2}{3} \left(\frac{dl}{\lambda}\right)^2 \sin^2(\omega t - \beta r)$$

The time-average power radiated by the dipole is given by

$$\langle P \rangle = \frac{2\pi \eta_0 I_0^2}{3} \left(\frac{dl}{\lambda}\right)^2 \langle \sin^2(\omega t - \beta r) \rangle$$

$$= \frac{\pi \eta_0 I_0^2}{3} \left(\frac{dl}{\lambda}\right)^2 .$$

$$= \frac{1}{2} I_0^2 \left[\frac{2\pi}{3} \eta_0 \left(\frac{dl}{\lambda}\right)^2 \right]$$

Thus the time-average power radiated by the dipole is the same as the time-average power dissipated in a resistance of value $\frac{2\pi\eta}{3}(dl/\lambda)^2$ when a current $I_0 \cos \omega t$ is passed through it. This is known as the "radiation resistance" and is denoted by the symbol R_{rad}. Thus for the Hertzian dipole,

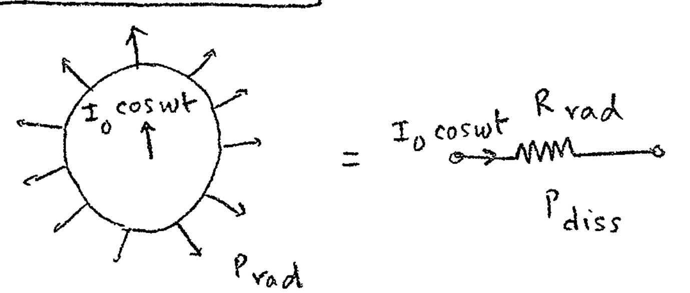
$$\boxed{R_{rad} = \frac{2\pi}{3}\eta_0 \left(\frac{dl}{\lambda}\right)^2 = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 \text{ ohms}}$$

An a numerical example,

for $\frac{dl}{\lambda} = 0.01$,

$R_{rad} = 80\pi^2 (0.01)^2 \approx 0.08$ ohms.

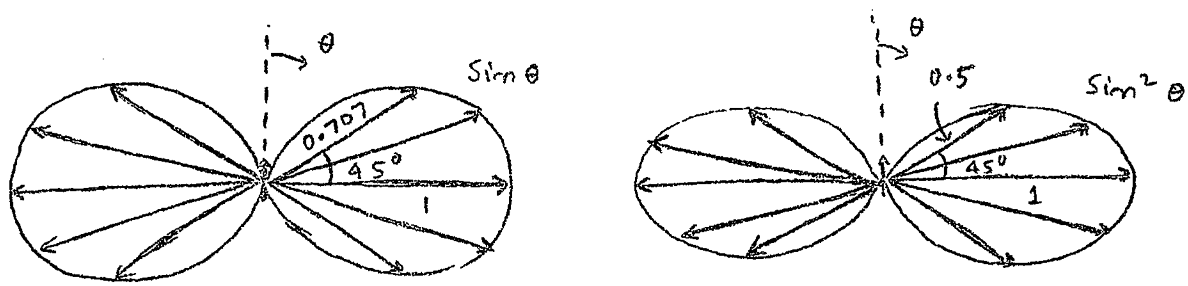
(not an effective radiator)



Let us now look at the directional characteristics of the radiation field of the Hertzian dipole. We note from the expressions for $\vec{E}$, $\vec{H}$ and $\vec{P}$ that

E_θ and $H_\phi \propto \sin\theta$ and $P_r \propto \sin^2\theta$

Thus the fields have a "radiation pattern" given by $\sin\theta$ whereas the power density radiation pattern is $\sin^2\theta$.



Field radiation pattern Power radiation pattern.

These radiation patterns indicate that the dipole radiation is directional, i.e., it radiates more effectively in certain directions than in certain other directions. In fact, it radiates strongest for $\theta = 90°$, i.e., in the plane normal to its axis and it does not radiate at all for $\theta = 0°$ and $180°$, i.e., along its axis.

We can now define a parameter known as the "<u>directivity</u>" of the antenna, denoted by the symbol D in the manner

$$D = \frac{\text{Maximum radiation intensity of the antenna}}{\text{Average radiation intensity of the antenna}}$$

What this means is that if we take the power radiated by the antenna and distribute it equally in all directions, then the average power density would be less than the maximum power density, i.e., the power density in the direction of maximum radiation. Obviously the more directional the radiation pattern is, the greater is the directivity. For the Hertzian dipole

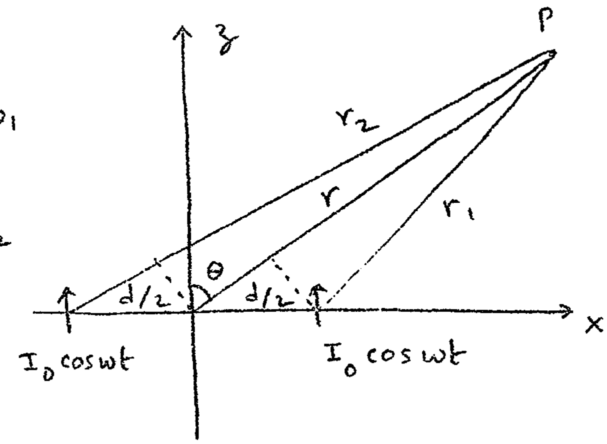
$$(P_r)_{max} = \frac{I_0^2 (dl)^2 \beta^2 \eta_0}{16 \pi^2 r^2} \sin^2(\omega t - \beta r)$$

$$(P_r)_{av} = \frac{I_0^2 (dl)^2 \beta^2 \eta_0}{16 \pi^2 \cdot 4\pi r^2} \sin^2(\omega t - \beta r) \cdot \frac{8\pi}{3}$$

$$= (P_r)_{max} \frac{2}{3}$$

$$\therefore \boxed{D = \frac{(P_r)_{max}}{(P_r)_{av}} = \frac{3}{2} = 1.5}$$

PRINCIPLE OF ANTENNA ARRAYS



$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$= -\frac{I\,dl\,\beta\eta_0 \sin\theta_1}{4\pi r_1}\sin(\omega t - \beta r_1)\,\vec{i}_{\theta_1}$$

$$-\frac{I\,dl\,\beta\eta_0 \sin\theta_2}{4\pi r_2}\sin(\omega t - \beta r_2)\,\vec{i}_{\theta_2}$$

$$\approx -\frac{I\,dl\,\beta\eta_0 \sin\theta}{4\pi r}$$

$$\cdot\left[\sin(\omega t - \beta r_1) + \sin(\omega t - \beta r_2)\right]\vec{i}_\theta$$

$$r_1 \approx r - \frac{d}{2}\sin\theta \quad , \quad r_2 \approx r + \frac{d}{2}\sin\theta$$

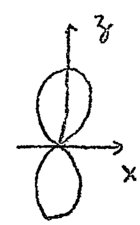
$$\sin(\omega t - \beta r_1) + \sin(\omega t - \beta r_2)$$

$$= \sin\left(\omega t - \beta r + \frac{\beta d}{2}\sin\theta\right) + \sin\left(\omega t - \beta r - \frac{\beta d}{2}\sin\theta\right)$$

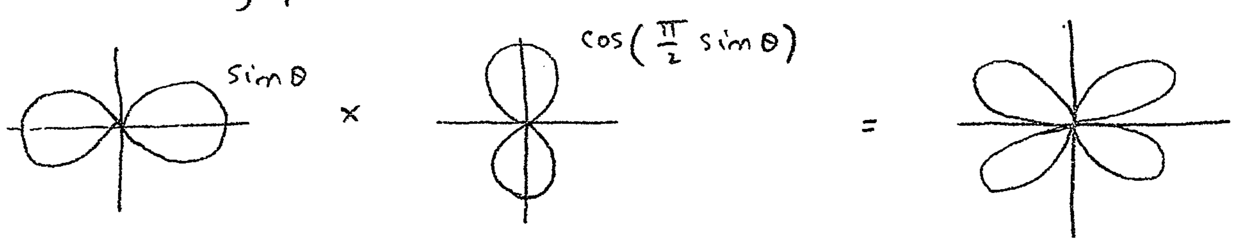
$$= 2\sin(\omega t - \beta r)\cos\frac{\beta d}{2}\sin\theta$$

$$\therefore E_\theta = -\frac{2\,I\,dl\,\beta\eta_0\sin\theta}{4\pi r}\underbrace{\cos\left(\frac{\beta d}{2}\sin\theta\right)}_{\text{array pattern}}\sin(\omega t - \beta r)$$

For $d = \frac{\lambda}{2}$, $\cos\left(\frac{\beta d}{2}\sin\theta\right) = \cos\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\sin\theta\right) = \cos\left(\frac{\pi}{2}\sin\theta\right) =$



The total pattern is given by the product of the unit pattern $\sin\theta$ and the array pattern.



HOMEWORK PROBLEM (Due 12/6/74)

A wire 1 m long is made to carry a uniform current of 10 amp at a frequency of 2 MHz.

a) Calculate the electric field intensity at a distance of 100 km in a direction at right angles to the axis of the wire.

b) Calculate the radiation resistance of the antenna.

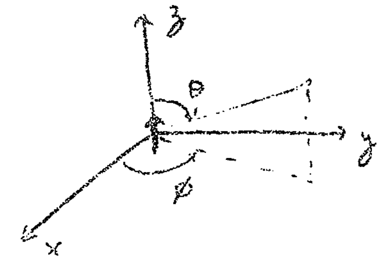
c) What is the average power radiated by the antenna.

HALF-WAVE DIPOLE

In the previous lecture, we found the radiation fields due to a Hertzian dipole located at the origin are given by

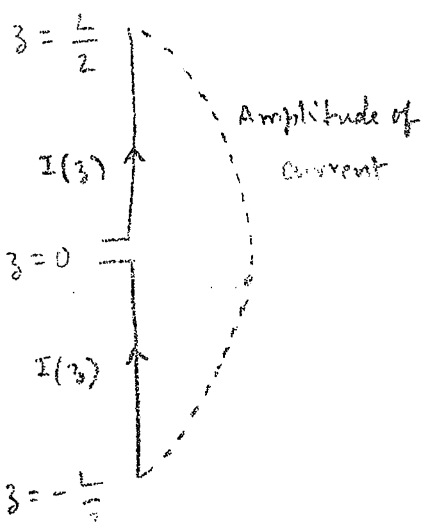
$$\overline{E} = -\frac{I_0\, dl\, \beta\, \eta_0 \sin\theta}{4\pi r}\sin(\omega t - \beta r)\ \vec{i}_\theta$$

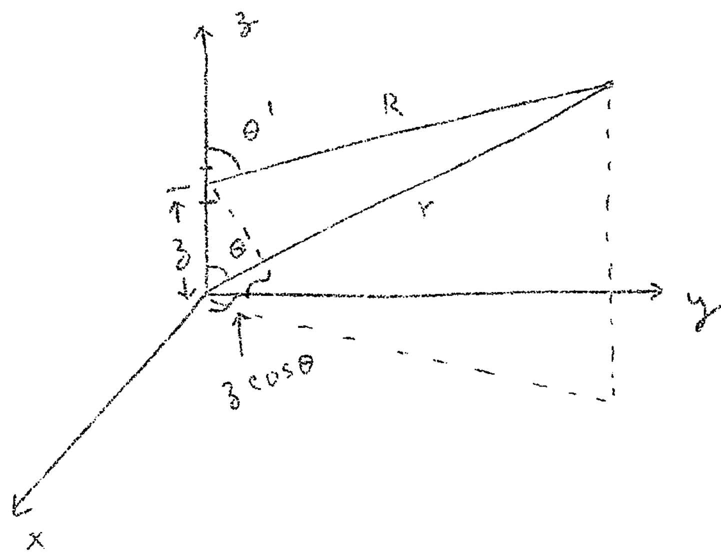
$$\vec{H} = -\frac{I_0\, dl\, \beta \sin\theta}{4\pi r}\sin(\omega t - \beta r)\ i_\phi$$



We shall use this result to find the radiation fields due to a half wave dipole. A half wave dipole is a straight wire antenna of length $L = \lambda/2$ and having a current distribution

$$I(z) = I_0 \cos\frac{\pi z}{L}\cos\omega t \quad \text{for } -\frac{L}{2} < z < \frac{L}{2}$$





Applying the results for the Hertzian dipole to the current distribution of the half wave dipole, we obtain

$$E_\theta = -\int_{z=-\frac{L}{2}}^{\frac{L}{2}} \frac{I_0 \cos\frac{\pi z}{L}\, dz\, \beta \eta_0 \sin\theta'}{4\pi R} \sin(\omega t - \beta R)$$

$$H_\phi = -\int_{z=-\frac{L}{2}}^{\frac{L}{2}} \frac{I_0 \cos\frac{\pi z}{L}\, dz\, \beta \sin\theta'}{4\pi R} \sin(\omega t - \beta R)$$

where R and θ' are functions of z.

For $R \gg L$, we can set $\theta' \approx \theta$ and $R \approx r$ in the amplitude factors in the integrands. For the R in the phase factor, however, we substitute $R \approx r - z\cos\theta$ since βR can vary appreciably for $-\frac{L}{2} < z < \frac{L}{2}$. Thus we have

$$E_\theta = -\int_{z=-\frac{L}{2}}^{\frac{L}{2}} \frac{I_0 \cos\frac{\pi z}{L}\, dz\, \beta \eta_0 \sin\theta}{4\pi r} \sin(\omega t - \beta r + \beta z\cos\theta)$$

$$= -\frac{I_0 \frac{\pi}{L} \eta_0 \sin\theta}{4\pi r} \int_{z=-\frac{L}{2}}^{\frac{L}{2}} \cos\frac{\pi z}{L} \sin\left(\omega t - \frac{\pi}{L} r + \frac{\pi}{L} z\cos\theta\right) dz$$

$$= -\frac{I_0 \eta_0 \sin\theta}{4 L r} \int_{z=-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{2}\left\{ \sin\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}(1+\cos\theta)\right] + \sin\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}(\cos\theta - 1)\right]\right\} dz$$

$$= + \frac{I_0 \eta_0 \sin\theta}{8LR} \left\{ \frac{\cos\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}(1+\cos\theta)\right]}{\frac{\pi}{L}(\cos\theta + 1)} + \frac{\cos\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}(\cos\theta - 1)\right]}{\frac{\pi}{L}(\cos\theta - 1)} \right\}_{z=-\frac{L}{2}}^{\frac{L}{2}}$$

$$= - \frac{I_0 \eta_0 \sin\theta}{4\pi r \sin^2\theta} \left\{ \frac{\pi}{L}\cos\theta \cdot \cos\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}\cos\theta\right] \cos\frac{\pi z}{L} + \sin\left[\omega t - \frac{\pi}{L} r + \frac{\pi z}{L}\cos\theta\right] \sin\frac{\pi z}{L} \right\}_{z=-\frac{L}{2}}^{\frac{L}{2}}$$

$$= - \frac{I_0 \eta_0}{4\pi r \sin\theta} \left\{ \sin\left(\omega t - \frac{\pi}{L} r + \frac{\pi}{2}\cos\theta\right) + \sin\left(\omega t - \frac{\pi}{L} r - \frac{\pi}{2}\cos\theta\right) \right\}$$

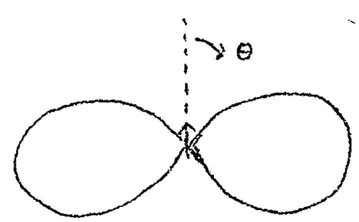
$$\boxed{E_\theta = - \frac{\eta_0 I_0}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \sin(\omega t - \beta r)}$$

similarly

$$\boxed{H_\phi = - \frac{I_0}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \sin(\omega t - \beta r)}$$

Thus the fields have radiation pattern given by $\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$.

θ	0°	30°	45°	60°	90°
$\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$	0	0.1747	0.3947	0.6665	1



The Poynting vector is given by

$$\vec{P} = \vec{E} \times \vec{H} = \frac{\eta_0 I_0^2}{4\pi^2 r^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin^2(\omega t - \beta r) \; \vec{i}_r$$

The power radiated by the antenna is given by

$$P = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \vec{P} \cdot r^2 \sin\theta \, d\theta \, d\phi \, \vec{i}_r$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\eta_0 I_0^2}{4\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \sin^2(\omega t - \beta r) \, d\theta \, d\phi$$

$$= \frac{\eta_0 I_0^2}{2\pi} \int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta \; \sin^2(\omega t - \beta r)$$

$$= \frac{\eta_0 I_0^2}{\pi} \int_{\theta=0}^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta \cdot \sin^2(\omega t - \beta r)$$

$$= \frac{\eta_0 I_0^2}{\pi} (0.609) \sin^2(\omega t - \beta r)$$

The time-average radiated power is given by

$$\langle P \rangle = \frac{\eta_0 I_0^2}{2\pi} (0.609) = \frac{1}{2} I_0^2 \left(\frac{0.609}{\pi} \eta_0 \right)$$

Thus

$$\boxed{R_{rad} = \frac{0.609 \, \eta_0}{\pi} = 0.609 \times 120 = 73 \text{ ohms.}}$$

The directivity is given by

$$D = \frac{\left[\dfrac{\eta_0 I_0^2}{4\pi^2 r^2} \dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \sin^2(\omega t - \beta r)\right]_{max}}{\left[\dfrac{\eta_0 I_0^2}{\pi}(0.609)\sin^2(\omega t - \beta r)\right] \Big/ 4\pi r^2}$$

$$= \frac{\dfrac{\eta_0 I_0^2}{4\pi^2 r^2}\sin^2(\omega t - \beta r)}{\dfrac{\eta_0 I_0^2}{4\pi^2 r^2}(0.609)\sin^2(\omega t - \beta r)}$$

$$= \frac{1}{0.609}$$

$$\boxed{D = 1.642}$$